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ON THE UNIQUENESS OF REPRESENTATION OF THE STRESS FIELD OF PLANE POLYGONAL DISLOCATION LOOPS

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Introduction

The theory of straight dislocation lines of infinite length can be applied only to a very limited class of problems, where the assumptions concerning the form of the dislocation are valid. However, in most situations of interest dislocations have geometries which do not allow the direct use of the results obtained for infinite lines. A relatively new geometrical approach to the problem was suggested by the results of Lothe (1) who deduced a simple expression for the force exerted at a point on one ray of an angular dislocation by the other ray. Brown (2) then gave a proof of Lothe's result and in doing so expressed the field of an arbitrary planar loop as a line integral along the loop considered. For an arbitrary planar loop or line segment, the field was shown to be a function of the zeroth and second derivatives of fields of infinitely straight dislocations lying in the same plane. Asaro and Barnett (3) reconsidered Brown's representation for the in-plane fields of planar dislocation loops. They showed that the Brown line integral can be partially integrated to yield expressions independent of the derivatives of the stress fields of infinitely straight dislocations. These expressions are particularly important for numerical computations since the infinite dislocation fields can be obtained to a much higher accuracy than those of the first and second derivatives, from the same steps in the numerical integration (4).

In this note the results of Asaro and Barnett are reconsidered with the intention of clarifying the nature of some apparent discontinuities that appear in the integrands of the line integrals involved, if certain assumptions concerning the features of the paths of integration are made. The uniqueness of representation of the stress field is thus established. Subsequently, the limitations of the resulting expressions are investigated. It is found that no complications arise with the existence of potential singularities and that the solution is well behaved regardless of the position of field points. Thus it is observed that for the case of "sharp" cornered polygonal paths of integration, the first partial integration of the Brown line integral gives rise to integrands that are discontinuous at the sharp corners. To avoid the discontinuities, the line integral is split into a finite sum of integrals, each of which is evaluated along each side of the polygon. The same problem is subsequently reanalyzed as a limiting case of a "smooth" cornered curve, at the corners of which the radius of curvature tends to zero. In the "smooth" corner approach no discontinuities appear in the line integrals involved. The two approaches are shown to yield the same result. The special features of the stress field that depend on the position of the field points are subsequently analyzed showing that the solution is well behaved at all limits.

Plane Polygonal Loops

A. "Sharp" corner approach:

The Brown (2) line integral representation for the stress field at a point $\underline{x}$ due to the existence of a dislocation loop L is given by:

$$\underline{\sigma}^L(\underline{x}) = 1/2\oint_L \frac{\underline{\Sigma}(\theta)+\underline{\Sigma}''(\theta)}{|\underline{x}-\underline{x}'|}\, d\theta \tag{1}$$

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where $\underline{x}'$ denotes any point on L. $\underline{\Sigma}(\theta)$ is the angular stress factor tensor for a straight dislocation lying along the direction $\underline{x}-\underline{x}'$ in the plane of L, whose Burger's vector is the same as the one of the loop. Let $\underline{t}$ be the unit tangent vector to L at $\underline{x}'$. Assume that L is a planar loop and that the field points under consideration lie in the same plane. Let angles α and θ be measured counterclockwise from a fixed datum to $\underline{t}$ and $\underline{x}-\underline{x}'$ respectively (see Fig. 1). Consider now the case of a closed polygonal loop L with N vertices. Let R_n denote the distance from $\underline{x}$ to the n^{th} vertex and let θ_n denote the direction of $|\underline{x}-\underline{x}'|_n$ relative to the fixed datum in the plane (see Fig. 2). The corners of the polygon are assumed to be perfectly "sharp". This is equivalent to saying that across the n^{th} vertex, the angle between the fixed datum and the tangent to the polygon changes discontinuously from α_n^- to α_n^+.

The integrand of Eq. (1) varies continuously along the loop. Thus by integration by parts and use of the fact that

$$\frac{d}{d\theta}|\underline{x}-\underline{x}'| = -|\underline{x}-\underline{x}'| \cot(\theta-\alpha) , \tag{2}$$

Eq. (1) gives:

$$\oint_L \frac{\underline{\Sigma}''(\theta)}{|\underline{x}-\underline{x}'|} d\theta = \left[\frac{\underline{\Sigma}'(\theta)}{|\underline{x}-\underline{x}'|}\right]_L - \oint_L \frac{\underline{\Sigma}'(\theta)}{|\underline{x}-\underline{x}'|} \cot(\theta-\alpha)\, d\theta \tag{3}$$

The first term of Eq. (3) vanishes. The integrand of the second term varies discontinuously across the vertices of the polygon since α changes discontinuously as each vertex is traversed. At this point, before a second partial integration is considered, the integral must be expressed in such a way that its integrand varies continuously along the integration path. To that effect the integral is considered as a finite sum of integrals each evaluated along each side of the polygon. Letting P_n^- and P_n^+ be points just to the right and left of the vertex n, Expression (3) can be written as:

$$\oint_L \frac{\underline{\Sigma}''(\theta)d\theta}{|\underline{x}-\underline{x}'|} = -\sum_{n=2}^{n=N}\int_{P_{n-1}^+}^{P_n^-} \frac{\underline{\Sigma}'(\theta)\cot(\theta-\alpha)d\theta}{|\underline{x}-\underline{x}'|} - \int_{P_N^+}^{P_1^-} \frac{\underline{\Sigma}'(\theta)\cot(\theta-\alpha)d\theta}{|\underline{x}-\underline{x}'|}$$

The integrands of the above N integrals are continuous and integration by parts can now be used giving:

$$\oint_L \frac{\underline{\Sigma}''(\theta)d\theta}{|\underline{x}-\underline{x}'|} = -\sum_{n=2}^{n=N}\left[\frac{\underline{\Sigma}(\theta)\cot(\theta-\alpha)}{|\underline{x}-\underline{x}'|}\right]_{P_{n-1}^+}^{P_n^-} - \left[\frac{\underline{\Sigma}(\theta)\cot(\theta-\alpha)}{|\underline{x}-\underline{x}'|}\right]_{P_N^+}^{P_1^-}$$

$$+\sum_{n=2}^{n=N}\int_{P_{n-1}^+}^{P_n^-} \underline{\Sigma}(\theta)\, d\left(\frac{\cot(\theta-\alpha)}{|\underline{x}-\underline{x}'|}\right) + \int_{P_N^+}^{P_1^-} \underline{\Sigma}(\theta)\, d\left(\frac{\cot(\theta-\alpha)}{|\underline{x}-\underline{x}'|}\right)$$

Rearranging terms in the above and using Eq. (2), expression (4) can be written as:

$$\frac{1}{2}\oint_L \frac{\underline{\Sigma}(\theta)+\underline{\Sigma}''(\theta)}{|\underline{x}-\underline{x}'|} d\theta = -\frac{1}{2}\sum_{n=1}^{n=N} \frac{\underline{\Sigma}(\theta_n)\cot(\theta_n-\alpha_n^-)}{|\underline{x}-\underline{x}'|_n} - \frac{1}{2}\sum_{n=2}^{n=N+1} \frac{\underline{\Sigma}(\theta_{n-1})\cot(\theta_{n-1}-\alpha_{n-1}^+)}{|\underline{x}-\underline{x}'|_{n-1}}$$

$$+\frac{1}{2}\sum_{n=2}^{n=N}\int_{P_{n-1}^+}^{P_n^-} \frac{\underline{\Sigma}(\theta)d\alpha}{|\underline{x}-\underline{x}'|\sin^2(\theta-\alpha)} + \frac{1}{2}\int_{P_N^+}^{P_1^-} \frac{\underline{\Sigma}(\theta)d\alpha}{|\underline{x}-\underline{x}'|\sin^2(\theta-\alpha)} \tag{4}$$

Observing that $d\alpha = 0$ along any side of the polygon, the two integral terms along the sides drop out. Setting $n = n + 1$ in the second of the sums of Eq. (4) the stress field is given by:

$$\underline{\sigma}^L(\underline{x}) = 1/2 \sum_{n=1}^{N} \frac{\underline{\Sigma}(\theta_n)\cot(\theta_n-\alpha)}{|\underline{x}-\underline{x}'|_n} \Bigg|_{\alpha = \alpha_n^-}^{\alpha = \alpha_n^+} \tag{5}$$

B. "Smooth" Corner approach:

For the case of a smooth planar loop L, Asaro and Barnett (3) showed that the stress field given in Eq. (1) can be reduced to:

$$\underline{\sigma}^L(x) = 1/2 \oint \frac{\underline{\Sigma}(\theta)d\alpha}{|\underline{x}-\underline{x}'|\sin^2(\theta-\alpha)} \tag{6}$$

The above formula was obtained after two successive partial integrations of the Brown integral. For smooth loops, α varies continuously along the line of integration and the problems of discontinuity encountered in the previous section are no longer present.

Assume now that the corners of the polygonal loop treated in the previous section are replaced by segments of circles (see dotted lines, Fig. 2). In this case, expression (6), which is valid for smooth curves, holds. Consider now the limit as the curvature of these circles tends to infinity. The contribution to Eq. (6) coming from the straight line parts of the sides will be zero since $d\alpha = 0$ along straight segments. Near each corner however, there will be a contribution to the field, because of the existence of the circular arcs. This will be given by:

$$\underline{\sigma}^L(\underline{x}) = \sum_{n=1}^{n=N} 1/2 \int_{\alpha_n^-}^{\alpha_n^+} \frac{\underline{\Sigma}(\theta)d\alpha}{|\underline{x}-\underline{x}'|\sin^2(\theta-\alpha)} \tag{7}$$

As the curvature of the arcs is increased ($R \rightarrow 0$), the values of θ and $|\underline{x}-\underline{x}'|$ along the arc approach θ_n and $|\underline{x}-\underline{x}|_n$ respectively. Thus Eq. (7) tends to:

$$\underline{\sigma}^L(x) = \sum_{n=1}^{N} \frac{\underline{\Sigma}(\theta_n)}{2|\underline{x}-\underline{x}'|_n} \int_{\alpha_n^-}^{\alpha_n^+} \frac{d\alpha}{\sin^2(\theta_n-\alpha)}$$

which is equal to

$$\underline{\sigma}^L(x) = \frac{1}{2} \sum_{n=1}^{N} \frac{\underline{\Sigma}(\theta_n)\cot(\theta_n-\alpha)}{|\underline{x}-\underline{x}'|_n} \Bigg|_{\alpha_n^-}^{\alpha_n^+} \tag{8}$$

Expressions (5) and (8) are as expected identical. Uniqueness of representation of the stress field is thus established.

Special Features of the Solution

In the previous sections, two different approaches were used to derive the expression for the stress field created by a planar polygonal loop. The result was obtained in the form of a finite sum of terms evaluated at the vertices. Examination of the solution shows that some of the terms of the sum might tend to infinity for certain values of $(\theta_n - \alpha_n^{\pm})$. In particular, if the field point $\underline{x}$ lies in the extension of the straight line joining the s and s+1 vertices, then:

$$\theta_s = \theta_{s+1},\ \alpha_s^+ - \theta_s = \pi \text{ and } \alpha_{s+1}^- - \theta_{s+1} = \pi .$$

In such a case, each of the following two terms of the sum become infinite.

$$\frac{\underline{\Sigma}(\theta_s)\cot(\theta_s-\alpha_s^+)}{|\underline{x}-\underline{x}'|_{s+1}} - \frac{\underline{\Sigma}(\theta_{s+1})\cot(\theta_{s+1}-\alpha_{s+1}^-)}{|\underline{x}-\underline{x}'|_{s+1}} \tag{9}$$

In the present section the behaviour of the solution at points collinear to the sides of the polygon will be investigated. In doing so, a field point $\underline{x}$ lying at a distance d from the extension of the side connecting vertices s and s+1 will be considered (see Fig. 3). From the geometry of the figure:

$$\cot(\theta_s-\alpha_s^+) = -\frac{|\underline{x}-\underline{x}'|_s}{d}\cos(\theta_s-\alpha_s^+), \quad \cot(\theta_{s+1}-\alpha_{s+1}^-) = -\frac{|\underline{x}-\underline{x}'|_{s+1}}{d}\cos(\theta_{s+1}-\alpha_{s+1}^-)$$

Using the above, expression (9) can be written as:

$$\frac{1}{d}\left[\underline{\Sigma}(\theta_{s+1})\cos(\theta_{s+1}-\alpha_{s+1}^-) - \underline{\Sigma}(\theta_s)\cos(\theta_s-\alpha_s^+)\right]$$
$$= \frac{(\theta_{s+1}-\omega)-(\theta_s-\omega)}{d}\left[\frac{\underline{\Sigma}(\theta_{s+1})\cos(\theta_{s+1}-\alpha_{s+1}^-)-\underline{\Sigma}(\theta_s)\cos(\theta_s-\alpha_s^+)}{\theta_{s+1}-\theta_s}\right] \tag{10}$$

where $\omega = \alpha_s^+ - \pi$.
Taking the limit as $d \to 0$, $\cos(\theta_s - \alpha_s^+) \to -1$, $\cos(\theta_{s+1} - \alpha_{s+1}^-) \to -1$ and $\frac{(\theta_{s+1}-\omega)}{d} \to \frac{1}{R_{s+1}}$ $\frac{(\theta_s-\omega)}{d} \to \frac{1}{R_s}$
Thus (10) becomes:

$$-\underline{\Sigma}'(\theta_s)\left[\frac{1}{R_{s+1}} - \frac{1}{R_s}\right] \tag{11}$$

Expression (11) is clearly nonsingular. For points collinear to consecutive vertices of the polygon, the field remains bounded, but the expression involves both the zeroth and first derivatives of $\underline{\Sigma}(\theta)$. The expression for the stress field must now be modified as follows:

$$\underline{\sigma}^L(\underline{x}) = \frac{1}{2}\sum_{\substack{n=1\\ n\neq s,\, s+1}}^{N} \frac{\underline{\Sigma}(\theta_n)\cot(\theta_n-\alpha)}{|\underline{x}-\underline{x}'|_n}\Bigg|_{\alpha_n^-}^{\alpha_n^+} - \frac{\underline{\Sigma}'(\theta_s)}{2}\left[\frac{1}{R_{s+1}} - \frac{1}{R_s}\right] + \frac{\underline{\Sigma}(\theta_s)}{2}\left[\frac{\cot\phi_s}{R_s} + \frac{\cot\phi_{s+1}}{R_{s+1}}\right] \tag{12}$$

where ϕ_s and ϕ_{s+1} are the interior angles of the polygon at the s^{th} and $s+1^{th}$ vertices.

At this point it is worth noting that any smooth loop can be considered as the common limit of a circumscribed and an inscribed polygonal loop whose number of sides tends to infinity. For such a case, $R_{s+1} \to R_s$ and the term of Eq. (12) involving the first derivative vanishes. This is expected since, as we have seen in (6), the field of a smooth loop can be expressed with respect to $\underline{\Sigma}(\theta)$ only, whatever the position of the field point $\underline{x}$ may be.

In conclusion, it is observed that there are no complications due to potential singularities arising for the case of field points collinear to two consecutive vertices of the polygonal loop. The stress field around a planar dislocation loop of any kind is found to be well behaved regardless of the position of the field point.

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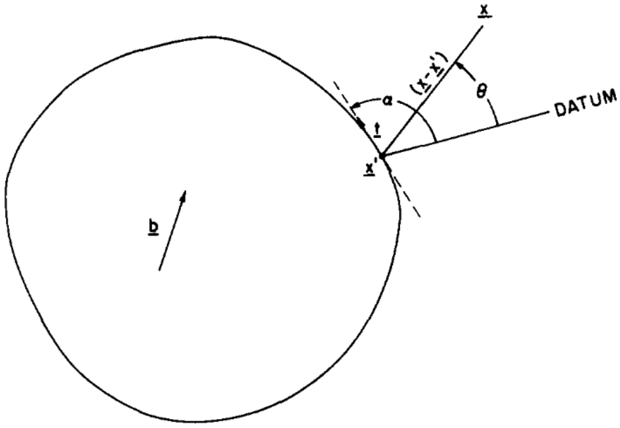

FIG. 1 Schematic of a smooth planar dislocation loop.

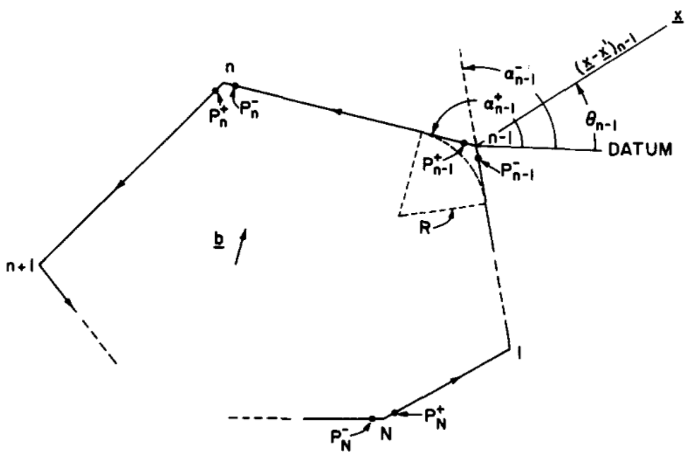

FIG. 2 A polygonal dislocation loop. Both the "smooth" and "sharp" corner approaches shown.

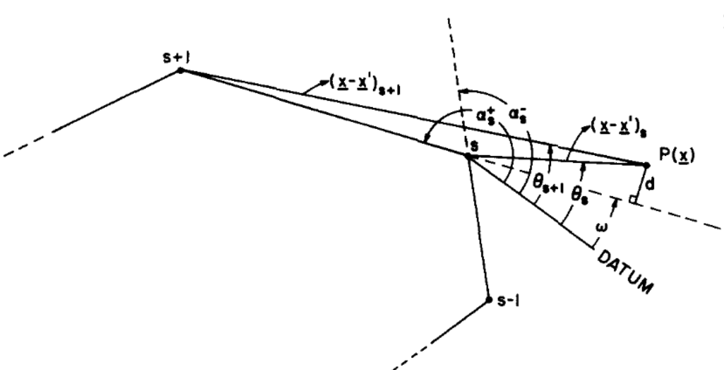

FIG. 3 Limiting case of a field point approaching the extension of a side of a polygonal dislocation loop.