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Three-dimensional elastostatics of a layer and a layered medium

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Abstract. This paper is concerned with the determination of the distribution of stresses and displacements in an infinite three-dimensional, linear, elastic, isotropic, homogeneous layer subjected to concentrated body forces acting upon an arbitrary internal point.

In §2 and §3 the governing partial differential field equations are reduced to a system or ordinary differential equations by the use of the two-dimensional Fourier transform, taken with respect to the two in-plane geometric variables (§4). Analytical expressions for the stresses and displacements are then obtained for the particular case of concentrated body forces, represented as Dirac delta functions (§5).

The results are subsequently utilized to formulate the multilayered medium problem by means of transfer matrices. In §8 the typical problem of a non-adhesive layered medium is undertaken.

1. General background

The problem of a single elastic layer in equilibrium has first been considered by Dougall (1904) [14], who conducted an extensive study of a thick plate subjected to arbitrary (surface or internal) loading using potential functions.

Multilayered half-space problems have received repeated attention because of their relevance to the theory of foundations, geotechnical engineering and composite materials. The well-known Boussinesq (1885) solution to the problem of a normal static load on the surface of a half-space [1] offers wide applications to loading problems in geophysics as well as various branches of engineering. This classic solution was extended to include the problems of normal static loads acting on the surface of a two and a three-layered half-space by Burminster (1945) [13] and Pario (1956) [21], of axisymmetrical loads by Harding and Sneddon (1945) [15,24], and of asymmetrical shear loads acting on the surface of a thick plate by Muki (1960) [20]. Also Kuo, in 1969 [17], obtained the solutions to the problem of inclined static loads on the surface of a multilayered medium through the Thomson-Haskell (1953) [16] matrix method using transfer matrices.

The same problem was later considered by Bufler, who provided a suitable and systematic matrix formulation in cartesian (1971) [9] and cylindrical (1974) [11] coordinates by means of two dimensional integral transforms introduced by Sneddon [23]. Bufler's papers contain a comprehensive elastostatic formulation of the multilayered system in the two-dimensional Fourier transform domain, but do not provide any information regarding the distribution of stresses and displacements in a single layer.

In this paper the more general problem of a three dimensional layer containing arbitrary internal loads is considered. For the specific case of a concentrated unit load, analytical expressions for the stresses and displacements are obtained in terms of convergent integrals. Representative stress components are plotted in Figs. 3, 4 and 5. The results are used in the formulation of the problem of the multilayer medium subjected to both surface and internal loads.

2. Field equations

The component forms of the balance law and constitutive equations of homogeneous, isotropic linear elastostatics are

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + F_x = 0, \tag{2.1a}$$

$$\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + F_y = 0, \tag{2.1b}$$

$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + F_z = 0, \tag{2.1c}$$

$$\sigma_{xx} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} u_{x,x} + \frac{\nu E}{(1+\nu)(1-2\nu)} (u_{y,y} + u_{z,z}), \tag{2.2a}$$

$$\sigma_{yy} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} u_{y,y} + \frac{\nu E}{(1+\nu)(1-2\nu)} (u_{x,x} + u_{z,z}), \tag{2.2b}$$

$$\sigma_{zz} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} u_{z,z} + \frac{\nu E}{(1+\nu)(1-2\nu)} (u_{x,x} + u_{y,y}), \tag{2.2c}$$

$$\sigma_{xz} = \frac{E}{2(1+\nu)} (u_{x,z} + u_{z,x}), \tag{2.2d}$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)} (u_{x,y} + u_{y,x}), \tag{2.2e}$$

$$\sigma_{yz} = \frac{E}{2(1+\nu)} (u_{y,z} + u_{z,y}), \tag{2.2f}$$

where x, y, z are spatial cartesian coordinates in an Euclidean 3-space; subscripts indicate components and a comma indicates partial differentiation with respect to the subscripted variable following it; E, ν are the Young's modulus of elasticity and Poisson's ratio respectively; F_x, F_y, F_z stand for the body force components.

Suitable combinations of the previous expressions, yield the following equations: From (2.1c),

$$\sigma_{zz,z} = -\sigma_{zx,x} - \sigma_{zy,y} - F_z;$$

from (2.1a), (2.2a), (2.2c) and (2.2e),

$$\sigma_{zx,z} = -\frac{E}{1-\nu^2}u_{x,xx} - \frac{E}{2(1-\nu)}u_{y,yx} - \frac{E}{2(1+\nu)}u_{x,yy} - \frac{\nu}{1-\nu}\sigma_{zz,x} - F_x;$$

from (2.1b), (2.2b), (2.2c) and (2.2e),

$$\sigma_{zy,z} = -\frac{E}{1-\nu^2}u_{y,yy} - \frac{E}{2(1-\nu)}u_{x,xy} - \frac{E}{2(1+\nu)}u_{y,xx} - \frac{\nu}{1-\nu}\sigma_{zz,y} - F_y;$$

from (2.2f),

$$u_{y,z} = \frac{2(1+\nu)}{E}\sigma_{zy} - u_{z,y};$$

from (2.2d),

$$u_{x,z} = \frac{2(1+\nu)}{E}\sigma_{zx} - u_{z,x};$$

from (2.2c),

$$u_{z,z} = \frac{(1+\nu)(1-2\nu)}{(1-\nu)E}\sigma_{zz} - \frac{\nu}{1-\nu}u_{x,x} - \frac{\nu}{1-\nu}u_{y,y};$$

from (2.2a), (2.2b) and (2.2c),

$$\sigma_{xx} + \sigma_{yy} = \frac{2\nu}{1-\nu}\sigma_{zz} + \frac{E}{1-\nu}(u_{x,x} + u_{y,y});$$

from (2.2a) and (2.2b),

$$\sigma_{xx} - \sigma_{yy} = \frac{E}{1+\nu}(u_{x,x} - u_{y,y});$$

and from (2.2e),

$$2\sigma_{xy} = \frac{E}{1+\nu}(u_{x,y} + u_{y,x}).$$

The first six expressions and the three last ones can be represented in matrix form as follows:

$$\frac{\partial \boldsymbol{a}}{\partial z} = \boldsymbol{A}\boldsymbol{a} + \boldsymbol{C}, \qquad (2.3)$$

$$\boldsymbol{b} = \boldsymbol{B}\boldsymbol{a}, \qquad (2.4)$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ define the column vectors

$$\boldsymbol{a} = (\sigma_{zz}, \sigma_{zx}, \sigma_{zy}, u_y, u_x, u_z)^T,$$

$$\boldsymbol{b} = (\sigma_{xx} + \sigma_{yy}, \sigma_{xx} - \sigma_{yy}, 2\sigma_{xy})^T,$$

where $(\;)^T$ stands for the transpose of a vector, and matrices the $\boldsymbol{A}$ and $\boldsymbol{B}$ are given by the equations on page 6.

$$
\boldsymbol{A} = \begin{pmatrix}
0 & -\dfrac{\partial}{\partial x} & -\dfrac{\partial}{\partial y} & 0 & 0 & 0 \\
-\dfrac{\nu}{1-\nu}\dfrac{\partial}{\partial x} & 0 & 0 & -\dfrac{E}{2(1-\nu)}\dfrac{\partial^2}{\partial x\,\partial y} & \left(-\dfrac{E}{2(1+\nu)}\dfrac{\partial^2}{\partial y^2} - \dfrac{E}{1-\nu^2}\dfrac{\partial^2}{\partial x^2}\right) & 0 \\
-\dfrac{\nu}{1-\nu}\dfrac{\partial}{\partial y} & 0 & 0 & \left(-\dfrac{E}{2(1+\nu)}\dfrac{\partial^2}{\partial x^2} - \dfrac{E}{1-\nu^2}\dfrac{\partial^2}{\partial y^2}\right) & -\dfrac{E}{2(1-\nu)}\dfrac{\partial^2}{\partial x\,\partial y} & 0 \\
0 & 0 & \dfrac{2(1+\nu)}{E} & 0 & 0 & -\dfrac{\partial}{\partial y} \\
0 & \dfrac{2(1+\nu)}{E} & 0 & 0 & 0 & -\dfrac{\partial}{\partial x} \\
\dfrac{(1+\nu)(1-2\nu)}{(1-\nu)E} & 0 & 0 & -\dfrac{\nu}{1-\nu}\dfrac{\partial}{\partial y} & -\dfrac{\nu}{1-\nu}\dfrac{\partial}{\partial x} & 0
\end{pmatrix},
$$

$$
\boldsymbol{B} = \begin{pmatrix}
\dfrac{2\nu}{1-\nu} & 0 & 0 & \dfrac{E}{1-\nu}\dfrac{\partial}{\partial y} & \dfrac{E}{1-\nu}\dfrac{\partial}{\partial x} & 0 \\
0 & 0 & 0 & -\dfrac{E}{1+\nu}\dfrac{\partial}{\partial y} & \dfrac{E}{1+\nu}\dfrac{\partial}{\partial x} & 0 \\
0 & 0 & 0 & \dfrac{E}{1+\nu}\dfrac{\partial}{\partial x} & \dfrac{E}{1+\nu}\dfrac{\partial}{\partial y} & 0
\end{pmatrix}.
$$

The column vector $\boldsymbol{C}$ being

$$\boldsymbol{C} = \left(-F_z, -F_x, -F_y, 0, 0, 0\right)^T.$$

3. Statement of the problem: Infinite layer

The matrix differential equation (2.3) relates the z-coordinate partial derivative of vector $\boldsymbol{a}$ with the vector $\boldsymbol{a}$ itself. The vector $\boldsymbol{a}$ is composed of the components of the tractions acting on a constant-z plane as well as the components of the displacements.

If x, y are the in-plane coordinates of the layer and z is the coordinate perpendicular to the faces, the matrix partial differential equation (2.3) can be transformed into an ordinary matrix differential equation by using the two-dimensional Fourier transform with respect to the coordinates x, y.

According to Bufler [9], the following geometric Fourier transforms are defined:

$$\bar{f}(\alpha, \beta) = \mathscr{F}[f(x, y)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y)\, \mathrm{e}^{i(\alpha x + \beta y)}\, \mathrm{d}x\, \mathrm{d}y;$$

$$\overset{\alpha}{\bar{f}}(\alpha, \beta) = \mathscr{F}_\alpha[f(x, y)] = \frac{1}{j_\alpha}\mathscr{F}[f(x, y)];$$

$$\overset{\beta}{\bar{f}}(\alpha, \beta) = \mathscr{F}_\beta[f(x, y)] = \frac{1}{j_\beta}\mathscr{F}[f(x, y)];$$

where

$$j_\alpha = i\frac{\alpha}{|\alpha|} \quad (\alpha \neq 0); \quad j_\alpha = i \quad (\alpha = 0),$$

$$j_\beta = i\frac{\beta}{|\beta|} \quad (\beta \neq 0); \quad j_\beta = i \quad (\beta = 0),$$

$$i = \sqrt{-1}.$$

The inverse transforms are

$$f(x, y) = \begin{pmatrix} \mathscr{F}^{-1}[\bar{f}(\alpha, \beta)] \\ \mathscr{F}_\alpha^{-1}[\bar{f}^\alpha(\alpha, \beta)] \\ \mathscr{F}_\beta^{-1}[\bar{f}^\beta(\alpha, \beta)] \end{pmatrix}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \begin{pmatrix} \bar{f}(\alpha, \beta) \\ j_\alpha \bar{f}^\alpha(\alpha, \beta) \\ j_\beta \bar{f}^\beta(\alpha, \beta) \end{pmatrix} \mathrm{e}^{-i(\alpha x + \beta y)}\, \mathrm{d}\alpha\, \mathrm{d}\beta, \tag{3.1}$$

and the transforms of the derivatives are given by

$$\left\{\begin{array}{l} \overset{\alpha}{\overline{\dfrac{\partial f}{\partial x}}} = |\alpha| \overset{\alpha}{\bar{f}} = -j_\alpha j_\beta |\alpha| \overset{\beta}{\bar{f}} \\ \overset{\alpha}{\overline{\dfrac{\partial f}{\partial x}}} = -|\alpha| \bar{f} \\ \overset{\alpha}{\overline{\dfrac{\partial^2 f}{\partial x^2}}} = -\alpha^2 \overset{\alpha}{\bar{f}} \\ \overset{\beta}{\overline{\dfrac{\partial^2 f}{\partial x^2}}} = -\alpha^2 \overset{\beta}{\bar{f}} \\ \overset{\alpha}{\overline{\dfrac{\partial^2 f}{\partial x\, \partial y}}} = -|\alpha|\,|\beta| \overset{\beta}{\bar{f}} \end{array}\right\}. \tag{3.2}$$

Here the function f is such that the following conditions are fulfilled

$$\left\{\begin{array}{llll} f(x, y) \to 0 & \text{as} & |x| \to \infty & \text{and} \quad |y| \to \infty \\ \dfrac{\partial f(x, y)}{\partial x} \to 0 & \text{as} & |x| \to \infty & \\ \dfrac{\partial f(x, y)}{\partial y} \to 0 & \text{as} & |y| \to \infty. & \end{array}\right\}.$$

By the application of the matrix operators

$$\boldsymbol{F} = \begin{pmatrix} \mathscr{F} & & & & & \\ & \mathscr{F}_\alpha & & & & \\ & & \mathscr{F}_\beta & & & \\ & & & \mathscr{F}_\beta & & \\ & & & & \mathscr{F}_\alpha & \\ & & & & & -\mathscr{F} \end{pmatrix} \quad \text{and } \boldsymbol{F}' = \begin{pmatrix} \mathscr{F} & & \\ & -\mathscr{F} & \\ & & -\mathscr{F} \end{pmatrix}$$

to (2.3), (2.4) and by using (3.2), these expressions yield

$$\frac{\mathrm{d}\bar{\boldsymbol{a}}}{\mathrm{d}z} = \bar{\boldsymbol{A}}\bar{\boldsymbol{a}} + \bar{\boldsymbol{C}}, \tag{3.3}$$

$$\bar{\boldsymbol{b}} = \bar{\boldsymbol{B}}\bar{\boldsymbol{a}}, \tag{3.4}$$

where

$$\bar{\boldsymbol{a}} = \boldsymbol{F}\boldsymbol{a} = \left(\bar{\sigma}_{zz}, \overset{\alpha}{\bar{\sigma}}_{zx}, \overset{\beta}{\bar{\sigma}}_{zy}, \overset{\beta}{\bar{u}}_y, \overset{\alpha}{\bar{u}}_x, \bar{u}_z\right)^T, \quad \bar{u}_z = \mathscr{F}[-u_z], \tag{3.5a}$$

$$\bar{\boldsymbol{b}} = \boldsymbol{F}'\boldsymbol{a} = \left(\bar{\sigma}_{xx} + \bar{\sigma}_{yy}, -\left[\bar{\sigma}_{xx} - \bar{\sigma}_{yy}\right], -2\bar{\sigma}_{xy}\right)^T, \tag{3.5b}$$

$$\bar{\boldsymbol{C}} = \boldsymbol{F}\boldsymbol{C} = \left(-\bar{F}_z, \overset{\alpha}{\bar{F}}_x, -\overset{\beta}{\bar{F}}_y, 0, 0, 0\right)^T, \tag{3.5c}$$

and the matrices $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{B}}$ are given by the equations on page 9.

$$\bar{A} = \begin{pmatrix} 0 & -|\alpha| & -|\beta| & 0 & 0 & 0 \\ \frac{\nu}{1-\nu}|\alpha| & 0 & 0 & \frac{E}{2(1-\nu)}|\alpha||\beta| & \left(\frac{E}{2(1+\nu)}\beta^2 + \frac{E}{1-\nu^2}\alpha^2\right) & 0 \\ \frac{\nu}{1-\nu}|\beta| & 0 & 0 & \left(\frac{E}{2(1+\nu)}\alpha^2 + \frac{E}{1-\nu^2}\beta^2\right) & \frac{E}{2(1-\nu)}|\alpha||\beta| & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 & -|\beta| \\ 0 & \frac{2(1+\nu)}{E} & 0 & 0 & 0 & -|\alpha| \\ \frac{-(1+\nu)(1-2\nu)}{(1-\nu)E} & 0 & 0 & \frac{\nu}{1-\nu}|\beta| & \frac{\nu}{1-\nu}|\alpha| & 0 \end{pmatrix},$$

$$\bar{B} = \begin{pmatrix} \frac{2\nu}{1-\nu} & 0 & 0 & \frac{E}{1-\nu}|\beta| & \frac{E}{1-\nu}|\alpha| & 0 \\ 0 & 0 & 0 & \frac{E}{1+\nu}|\beta| & -\frac{E}{1+\nu}|\alpha| & 0 \\ 0 & 0 & 0 & \frac{E}{1+\nu}|\alpha|\, j_\alpha j_\beta & \frac{E}{1+\nu}|\beta|\, j_\alpha j_\beta & 0 \end{pmatrix}.$$

$$\bar{A}^* = \begin{pmatrix} 0 & -\frac{\alpha^*}{h} & -\frac{\beta^*}{h} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu}\frac{\alpha^*}{h} & 0 & 0 & \frac{1}{2(1-\nu)}\frac{E}{E^*}\frac{\alpha^*\beta^*}{h} & \frac{1}{1-\nu^2}\frac{E}{E^*}\left(\frac{2\alpha^{*2}+(1-\nu)\beta^{*2}}{2h}\right) & 0 \\ \frac{\nu}{1-\nu}\frac{\beta^*}{h} & 0 & 0 & \frac{1}{1-\nu^2}\frac{E}{E^*}\left(\frac{2\beta^{*2}+(1-\nu)\alpha^{*2}}{2h}\right) & \frac{1}{2(1-\nu)}\frac{E}{E^*}\frac{\alpha^*\beta^*}{h} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{h}\frac{E^*}{E} & 0 & 0 & -\frac{\beta^*}{h} \\ 0 & \frac{2(1+\nu)}{h}\frac{E^*}{E} & 0 & 0 & 0 & -\frac{\alpha^*}{h} \\ \frac{-(1+\nu)(1-2\nu)}{(1-\nu)h}\frac{E^*}{E} & 0 & 0 & \frac{\nu}{1-\nu}\frac{\beta^*}{h} & \frac{\partial}{1-\nu}\frac{\alpha^*}{h} & 0 \end{pmatrix},$$

$$\bar{B}^* = \begin{pmatrix} \frac{2\nu}{1-\nu} & 0 & 0 & \frac{1}{1-\nu}\frac{E}{E^*}\beta^* & \frac{1}{1-\nu}\frac{E}{E^*}\alpha^* & 0 \\ 0 & 0 & 0 & \frac{1}{1+\nu}\frac{E}{E^*}\beta^* & -\frac{1}{1+\nu}\frac{E}{E^*}\alpha^* & 0 \\ 0 & 0 & 0 & \frac{1}{1+\nu}\frac{E}{E^*}\alpha^* j_\alpha j_\beta & \frac{1}{1+\nu}\frac{E}{E^*}\beta^* j_\alpha j_\beta & 0 \end{pmatrix},$$

According to Bufler [9], we shall define normalized displacements as $u_i^* = \frac{E^*}{h} u_i$, where h stands for the thickness of the layer and E^* is a reference Young modulus introduced for normalization purposes to be used in the multilayered problem. Similarly, the dimensionless transform parameters are defined as:

$$\alpha^* = |\alpha| h, \quad \beta^* = |\beta| h, \quad \lambda = \sqrt{\alpha^{*2} + \beta^{*2}}.$$

Thus, equations (3.3), (3.4) will give

$$\frac{\mathrm{d}\bar{\boldsymbol{a}}^*}{\mathrm{d}z} = \boldsymbol{A}^* \bar{\boldsymbol{a}}^* + \bar{\boldsymbol{C}}, \tag{3.6}$$

$$\bar{\boldsymbol{b}} = \bar{\boldsymbol{B}}^* \bar{\boldsymbol{a}}^*, \tag{3.7}$$

where the new matrices $\bar{\boldsymbol{A}}^*$ and $\bar{\boldsymbol{B}}^*$ are given by the equations on page 10, and the vector $\bar{\boldsymbol{a}}^*$ is given by

$$\bar{\boldsymbol{a}}^* = \left(\bar{\sigma}_{zz}, \overset{\alpha}{\bar{\sigma}}_{zx}, \overset{\beta}{\bar{\sigma}}_{zy}, \overset{\beta}{\bar{u}}, \overset{\alpha}{\bar{u}}_x^*, \bar{u}_z^* \right)^T,$$

and will be refered to as the *state vector*.

4. Matrix differential equation

Equation (3.6) is an ordinary matrix differential equation which can be solved using the Cayley-Hamilton theorem [2].

For an arbitrary point at a distance z from the lower surface of the layer, the *state vector* is given by

$$\bar{\boldsymbol{a}}^*(z) = \boldsymbol{X}(z) \boldsymbol{X}^{-1}(0) \bar{\boldsymbol{a}}^*(0) + \boldsymbol{X}(z) \int_0^z \boldsymbol{X}^{-1}(s) \bar{\boldsymbol{C}}(s) \, \mathrm{d}s, \tag{4.1}$$

where $\bar{\boldsymbol{a}}^*(0)$ represents the initial value of $\bar{\boldsymbol{a}}^*$ at $z = 0$, $\boldsymbol{X}(z)$ is the fundamental matrix defined by the matrix of eigenvectors of $\bar{\boldsymbol{A}}^*$ postmultiplied by the matrix of eigenvalues of $\bar{\boldsymbol{A}}^*$ and s is a dummy variable.

a. Matrix of eigenvalues

The matrix of eigenvalues is a diagonal matrix whose elements are the exponentials of the eigenvalues of $\bar{\boldsymbol{A}}^*$. $\bar{\boldsymbol{A}}^*$ has two eigenvalues, $\frac{\lambda}{h}$ and $-\frac{\lambda}{h}$, each with a multiplicity order of three. The matrix of eigenvalues is given by

$$
\boldsymbol{Ev}(z) \equiv \begin{pmatrix} e^{\frac{\lambda}{h}z} & & & & & \\ & e^{\frac{\lambda}{h}z} & & & & \\ & & e^{-\frac{\lambda}{h}z} & & & \\ & & & e^{-\frac{\lambda}{h}z} & & \\ & & & & e^{\frac{\lambda}{h}z} & \\ & & & & & e^{-\frac{\lambda}{h}z} \end{pmatrix},
$$

where

$$
\lambda = \sqrt{\alpha^{*2} + \beta^{*2}}\,.
$$

b. Matrix of eigenvectors

The columns of this matrix are the eigenvectors of $\bar{A}^*$. The first and second columns and the third and fourth ones are the eigenvectors corresponding to the first and second eigenvalues, which have a multiplicity order of three. The fifth and sixth columns are the generalized eigenvectors of the first and second eigenvalues, respectively,

$$
\boldsymbol{Fm} \equiv [a_{ij}], \tag{4.2}
$$

where

$$
a_{11} = 0, \quad a_{12} = -2\frac{E}{E^*}\lambda\beta^*, \quad a_{13} = 0, \quad a_{14} = 2\frac{E}{E^*}\lambda\beta^*,
$$

$$
a_{15} = -\frac{E}{E^*}\lambda^2\alpha^* z, \quad a_{16} = -\frac{E}{E^*}\lambda^2\alpha^* z, \quad a_{21} = \frac{E}{E^*}\lambda\beta^*, \quad a_{22} = \frac{E}{E^*}\alpha^*\beta^*,
$$

$$
a_{23} = -\frac{E}{E^*}\lambda\beta^*, \quad a_{24} = \frac{E}{E^*}\alpha^*\beta^*, \quad a_{25} = \frac{E}{E^*}(\lambda^2 h + \lambda\alpha^{*2} z - \beta^2 \nu h),
$$

$$
a_{26} = \frac{E}{E^*}(\lambda^2 h - \lambda\alpha^{*2} z - \beta^2 \nu h), \quad a_{31} = -\frac{E}{E^*}\lambda\alpha^*,
$$

$$
a_{32} = \frac{E}{E^*}(\lambda^2 + \beta^{*2}), \quad a_{33} = \frac{E}{E^*}\lambda\alpha^*, \quad a_{34} = \frac{E}{E^*}(\lambda^2 + \beta^{*2}),
$$

$$
a_{35} = \frac{E}{E^*}\alpha^*\beta^*(\lambda z + \nu h), \quad a_{36} = -\frac{E}{E^*}\alpha^*\beta^*(\lambda z - \nu h),
$$

$$
a_{41} = -2(1+\nu)\alpha^*, \quad a_{42} = 2(1+\nu)\lambda, \quad a_{43} = -2(1+\nu)\alpha^*,
$$

$$
a_{44} = -2(1+\nu)\lambda, \quad a_{45} = (1+\nu)\alpha^*\beta^* z, \quad a_{46} = (1+\nu)\alpha^*\beta^* z,
$$

$$
a_{51} = 2(1+\nu)\beta^*, \quad a_{52} = 0, \quad a_{53} = 2(1+\nu)\beta^*, \quad a_{54} = 0,
$$

$$a_{55} = (1+\nu)\left[\alpha^{*2} z + 2(1-\nu)\lambda h\right], \quad a_{56} = (1+\nu)\left[\alpha^{*2} z - 2(1-\nu)\lambda h\right],$$
$$a_{61} = 0, \quad a_{62} = 2(1+\nu)\beta^{*}, \quad a_{63} = 0, \quad a_{64} = 2(1+\nu)\beta^{*},$$
$$a_{65} = (1+\nu)\alpha^{*}\left[\lambda z + (2\nu - 1)/h\right], \quad a_{66} = -(1+\nu)\alpha^{*}\left[\lambda z + (1-2\nu)h\right].$$

c. *Transfer matrix*

This matrix is given by $X(z)X^{-1}(0)$, and relates the *state vector* $\bar{a}^{*}(z)$ of any arbitrary point z with the *initial vector* $\bar{a}^{*}(0)$.

We shall denote it by

$$T(z) \equiv X(z)X^{-1}(0) = \frac{1}{2(1-\nu)}\cosh\left(\lambda\frac{z}{h}\right)\left[t_{ij}\right], \tag{4.3}$$

where

$$t_{11} = 2(1-\nu) - \lambda\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right), \quad t_{12} = -\frac{\alpha^{*}}{\lambda}\left[\lambda\frac{z}{h} + (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right],$$
$$t_{13} = -\frac{\beta^{*}}{\lambda}\left[\lambda\frac{z}{h} + (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right],$$
$$t_{14} = -\frac{E}{E^{*}}\frac{1}{1+\nu}\lambda\beta^{*}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right),$$
$$t_{15} = -\frac{E}{E^{*}}\frac{1}{1+\nu}\lambda\alpha^{*}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right), \quad t_{16} = \frac{E}{E^{*}}\frac{1}{1+\nu}\lambda\left[\lambda\frac{z}{h} - \tanh\left(\lambda\frac{z}{h}\right)\right],$$
$$t_{21} = \frac{\alpha^{*}}{\lambda}\left[\lambda\frac{z}{h} - (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right], \quad t_{22} = 2(1-\nu) + \frac{\alpha^{*2}}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right),$$
$$t_{23} = \frac{\alpha^{*}\beta^{*}}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right), \quad t_{24} = \frac{E}{E^{*}}\frac{1}{1+\nu}\frac{\alpha^{*}\beta^{*}}{\lambda}\left[\lambda\frac{z}{h} = \nu\tanh\left(\lambda\frac{z}{h}\right)\right],$$
$$t_{25} = \frac{E}{E^{*}}\frac{1}{1+\nu}\lambda\left[\frac{\alpha^{*2}}{\lambda}\frac{z}{h} + \left(1 - \nu\frac{\beta^{*2}}{\lambda^{2}}\right)\tanh\left(\lambda\frac{z}{h}\right)\right],$$
$$t_{26} = -\frac{E}{E^{*}}\frac{1}{1+\nu}\lambda\alpha^{*}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right),$$
$$t_{31} = \frac{\beta^{*}}{\lambda}\left[\lambda\frac{z}{h} - (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right], \quad t_{32} = \frac{\alpha^{*}\beta^{*}}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right),$$
$$t_{33} = 2(1-\nu) + \frac{\beta^{*2}}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right),$$
$$t_{34} = \frac{E}{E^{*}}\frac{1}{1+\nu}\lambda\left[\frac{\beta^{*2}}{\lambda}\frac{z}{h} + \left(1 - \nu\frac{\alpha^{*2}}{\lambda^{2}}\right)\tanh\left(\lambda\frac{z}{h}\right)\right],$$
$$t_{35} = \frac{E}{E^{*}}\frac{1}{1+\nu}\frac{\alpha^{*}\beta^{*}}{\lambda}\left[\lambda\frac{z}{h} + \nu\tanh\left(\lambda\frac{z}{h}\right)\right],$$

$$t_{36} = -\frac{E}{E^*}\frac{1}{1+\nu}\lambda\beta^*\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right), \quad t_{41} = \frac{E^*}{E}(1+\nu)\frac{\beta^*}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right),$$

$$t_{42} = \frac{E^*}{E}(1+\nu)\frac{\alpha^*\beta^*}{\lambda^2}\left[\frac{z}{h} - \frac{1}{\lambda}\tanh\left(\lambda\frac{z}{h}\right)\right],$$

$$t_{43} = \frac{E^*}{E}(1+\nu)\frac{1}{\lambda}\left\langle 4(1-\nu)\tanh\left(\lambda\frac{z}{h}\right) - \frac{\beta^{*2}}{\lambda^2}\left[\tanh\left(\lambda\frac{z}{h}\right) - \lambda\frac{z}{h}\right]\right\rangle,$$

$$t_{44} = 2(1-\nu) + \frac{\beta^{*2}}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right),$$

$$t_{45} = \frac{\alpha^*\beta^*}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right), \quad t_{46} = -\frac{\beta^*}{\lambda}\left[\lambda\frac{z}{h} + (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right],$$

$$t_{51} = \frac{E^*}{E}(1+\nu)\frac{\alpha^*}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right),$$

$$t_{52} = \frac{E^*}{E}(1+\nu)\frac{1}{\lambda}\left\langle 4(1-\nu)\tanh\left(\lambda\frac{z}{h}\right) - \frac{\alpha^{*2}}{\lambda^2}\left[\tanh\left(\lambda\frac{z}{h}\right) - \lambda\frac{z}{h}\right]\right\rangle,$$

$$t_{53} = \frac{E^*}{E}(1+\nu)\frac{\alpha^*\beta^*}{\lambda^2}\left[\frac{z}{h} - \frac{1}{\lambda}\tanh\left(\lambda\frac{z}{h}\right)\right], \quad t_{54} = \frac{\alpha^*\beta^*}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right),$$

$$t_{55} = 2(1-\nu) + \frac{\alpha^{*2}}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right),$$

$$t_{56} = -\frac{\alpha^*}{\lambda}\left[\lambda\frac{z}{h} + (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right],$$

$$t_{61} = \frac{E^*}{E}(1+\nu)\left[\frac{z}{h} - \frac{3-4\nu}{\lambda}\tanh\left(\lambda\frac{z}{h}\right)\right],$$

$$t_{62} = \frac{E^*}{E}(1+\nu)\frac{\alpha^*}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right),$$

$$t_{63} = \frac{E^*}{E}(1+\nu)\frac{\beta^*}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right), \quad t_{64} = \frac{\beta^*}{\lambda}\left[\lambda\frac{z}{h} - (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right],$$

$$t_{65} = \frac{\alpha^*}{\lambda}\left[\lambda\frac{z}{h} - (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right], \quad t_{66} = 2(1-\nu) - \lambda\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right).$$

e. Flexibility matrices

Expression (4.1), with the aid of (4.3), can be written as

$$\bar{a}^*(z) = T(z)\bar{a}^*(0) + \bar{R}(z), \tag{4.4}$$

where $\bar{R}^*(z) = X(z)\int_0^z X^{-1}(s)\bar{C}(s)\,ds$.

The *state vector* $\bar{a}^*$ is composed of stresses and displacements in the following way

$$\bar{a}^*(z) = \begin{pmatrix} \bar{\sigma}(z) \\ \bar{u}^*(z) \end{pmatrix},$$

where

$$\bar{\sigma}(z)=\left(\bar{\sigma}_{zz},\ \overset{\alpha}{\bar{\sigma}}_{zx},\ \overset{\beta}{\bar{\sigma}}_{zy}\right)^T,\quad \bar{\boldsymbol{u}}^*(z)=\left(\overset{\beta}{\bar{u}}{}_y^*,\ \overset{\alpha}{\bar{u}}{}_x^*,\ \bar{u}_z^*\right)^T.$$

If we are interested in relating displacements with stresses; e.g. stresses are known by the boundary conditions of our problem; this can be done using expression (4.4) in the form:

$$\begin{pmatrix}\bar{\sigma}(z)\\ \bar{\boldsymbol{u}}^*(z)\end{pmatrix}=\begin{pmatrix}\boldsymbol{T}_{11}(z) & \boldsymbol{T}_{12}(z)\\ \boldsymbol{T}_{21}(z) & \boldsymbol{T}_{22}(z)\end{pmatrix}\begin{pmatrix}\bar{\sigma}(0)\\ \bar{\boldsymbol{u}}^*(0)\end{pmatrix}+\begin{pmatrix}\overline{\boldsymbol{R}}_1(z)\\ \overline{\boldsymbol{R}}_2(z)\end{pmatrix},$$

where $\boldsymbol{T}_{ij}(z)$ ($\forall i,\ j=1, 2$) stand for the submatrices of $\boldsymbol{T}(z)$ defined in (4.3) and $\overline{\boldsymbol{R}}_i(z)$ are two column vectors containing the first three and the last three components of $\overline{\boldsymbol{R}}(z)$ respectively.

From the former expression,

$$\begin{pmatrix}\bar{\boldsymbol{u}}^*(0)\\ \bar{\boldsymbol{u}}^*(z)\end{pmatrix}=\begin{pmatrix}-\boldsymbol{T}_{12}^{-1}(z)\boldsymbol{T}_{11}(z) & \boldsymbol{T}_{12}^{-1}(z)\\ \boldsymbol{T}_{21}(z)-\boldsymbol{T}_{22}(z)\boldsymbol{T}_{12}^{-1}(z)\boldsymbol{T}_{11}(z) & \boldsymbol{T}_{22}(z)\boldsymbol{T}_{12}^{-1}(z)\end{pmatrix}\begin{pmatrix}\bar{\sigma}(0)\\ \bar{\sigma}(z)\end{pmatrix}$$
$$+\begin{pmatrix}-\boldsymbol{T}_{12}^{-1}(z) & 0\\ -\boldsymbol{T}_{22}(z)\boldsymbol{T}_{12}^{-1}(z) & 1\end{pmatrix}\begin{pmatrix}\overline{\boldsymbol{R}}_1(z)\\ \overline{\boldsymbol{R}}_2(z)\end{pmatrix}. \tag{4.5}$$

f. Internal point state vector

For an arbitrary point, inside the layer, equation (4.4) holds:

$$\bar{\boldsymbol{a}}^*(z)=\boldsymbol{T}(z)\bar{\boldsymbol{a}}^*(0)+\overline{\boldsymbol{R}}(z). \tag{4.6}$$

Also, for $z=h$, from (4.5) we obtain

$$\bar{\boldsymbol{u}}^*(0)=-\boldsymbol{T}_{12}^{-1}(h)\boldsymbol{T}_{11}(h)\bar{\sigma}(0)+\boldsymbol{T}_{12}^{-1}(h)\bar{\sigma}(h)-\boldsymbol{T}_{12}^{-1}(h)\overline{\boldsymbol{R}}_1(h).$$

Substituting $\bar{\boldsymbol{u}}^*(0)$ from above, into (4.6), we get

$$\begin{pmatrix}\bar{\sigma}(z)\\ \bar{\boldsymbol{u}}^*(z)\end{pmatrix}=\begin{pmatrix}\boldsymbol{T}_{11}(z)-\boldsymbol{T}_{12}(z)\boldsymbol{T}_{12}^{-1}(h)\boldsymbol{T}_{11}(h) & \boldsymbol{T}_{12}(z)\boldsymbol{T}_{12}^{-1}(h)\\ \boldsymbol{T}_{21}(z)-\boldsymbol{T}_{22}(z)\boldsymbol{T}_{12}^{-1}(h)\boldsymbol{T}_{11}(h) & \boldsymbol{T}_{22}(z)\boldsymbol{T}_{12}^{-1}(h)\end{pmatrix}\begin{pmatrix}\bar{\sigma}(0)\\ \bar{\sigma}(h)\end{pmatrix}$$
$$+\begin{pmatrix}\boldsymbol{T}_{12}(z)\boldsymbol{T}_{12}^{-1}(h)\overline{\boldsymbol{R}}_1(h)+\overline{\boldsymbol{R}}_1(z)\\ -\boldsymbol{T}_{22}(z)\boldsymbol{T}_{12}^{-1}(h)\overline{\boldsymbol{R}}_1(h)+\overline{\boldsymbol{R}}_2(z)\end{pmatrix}, \tag{4.7}$$

which represents the tranforms of the stresses and displacements of any arbitrary point with respect to the transforms of the tractions on the surfaces and the transform of the applied body forces.

5. Nonhomogeneous system: Layer with concentrated body forces

In this section we shall consider the solution of a layer subjected to concentrated forces of unit magnitude acting in an arbitrary direction and applied to any internal point.

Let $\xi(0, 0, H)$ be the point where the force is applied and $X(x, y, z)$ be the point of observation, as depicted in Fig. 1. If $\delta(x, y, z)$ stands for the Dirac delta function defined in the geometric domain. Arbitrary forces in the three directions will be expressed as

$$F^x = (\delta(x, y, z-H)e_x, 0, 0)^T, \quad F^y = (0, \delta(x, y, z-H)e_y, 0)^T,$$
$$F^z = (0, 0, \delta(x, y, z-H)e_z)^T.$$

The transformed expressions for the body forces are given by

$$\overset{\alpha}{\bar{F}^x} = \frac{1}{2\pi}\frac{1}{j_\alpha}\delta(z-H)e_x, \quad \overset{\beta}{\bar{F}^y} = \frac{1}{2\pi}\frac{1}{j_\beta}\delta(z-H)e_y,$$
$$\bar{F}^z = \frac{1}{2\pi}\delta(2-H)e_z, \tag{5.1}$$

where e_i is the unit vector in the i-th direction.

By applying, sequentially, expressions (5.1) to the second term of the right hand side of (4.1) or (4.4), we get

$$\bar{R}^x(z) = X(z)X^{-1}(H)\left(0, \frac{1}{2\pi}\frac{1}{j_\alpha}, 0, 0, 0, 0\right)^T,$$

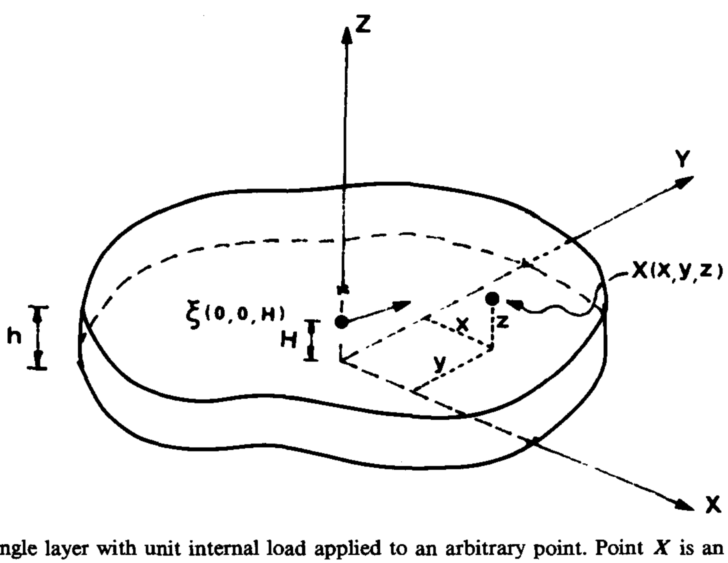

Fig. 1. Single layer with unit internal load applied to an arbitrary point. Point X is an arbitrary internal point.

$$\overline{R}^y(z) = X(z)X^{-1}(H)\left(0, 0, -\frac{1}{2\pi}\frac{1}{j_\beta}, 0, 0, 0\right)^T,$$

$$\overline{R}^z(z) = X(z)X^{-1}(H)\left(-\frac{1}{2\pi}, 0, 0, 0, 0, 0\right)^T,$$

or see eqns. (5.2a)–(5.2c) on pages 18–19, where the superscript (x, y, z) denotes the direction of the unit load.

a. Internal point state vector

In this particular case of concentrated body forces, we shall write

$$\bar{\sigma}(0) = \bar{\sigma}(h) = 0.$$

It follows from (4.7) that

$$\begin{pmatrix} \bar{\sigma}(z) \\ \bar{u}^*(z) \end{pmatrix} = \begin{pmatrix} -T_{12}(z)T_{12}^{-1}(h)\overline{R}_1(h) + \overline{R}_1(z) \\ -T_{22}(z)T_{12}^{-1}(h)\overline{R}_1(h) + \overline{R}_2(z) \end{pmatrix},$$

for any point z above the load level H $(z \geqslant H)$, otherwise the additional terms $\overline{R}_1(z)$ and $\overline{R}_2(z)$ should be dropped.

Thus, the former expression, for any arbitrary point z, gives

$$\begin{pmatrix} \bar{\sigma}(z) \\ \bar{u}^*(z) \end{pmatrix} = \begin{pmatrix} -T_{12}(z)T_{12}^{-1}(h)\overline{R}_1(h) + \overline{R}_1(z)\mathscr{H}(z-H) \\ -T_{22}(z)T_{12}^{-1}(h)\overline{R}_1(h) + \overline{R}_2(z)\mathscr{H}(z-H) \end{pmatrix}, \tag{5.3}$$

with

$$\mathscr{H}(z-H) = \begin{cases} 1, & \forall z \geqslant H; \\ 0, & \forall z < H. \end{cases}$$

b. Analytical expressions for the stresses and displacements

Letting superscripts (x, y, z) denote the direction of the unit load and setting $\frac{z}{h} = \chi$, $\frac{H}{h} = \psi$, we find from equations (3.1), (5.3) and Appendix A that

$$\sigma_{zz}^z(x, y, z) = \frac{1}{4\pi(1-\nu)h^2}\int_{\lambda=0}^{\lambda=\infty} f_{zz}^z(\lambda) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda, \tag{5.4}$$

where

$$f_{zz}^z = \frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{\lambda^4\chi(1-\psi)\sinh[\lambda(\chi-\psi)]$$

$$+\frac{\lambda^3}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)] + ((4\nu-3)\chi+\psi-1)$$

$$\overline{R}^{x}(z)=\frac{1}{4\pi(1-\nu)\lambda j_{\alpha}}\left\{\begin{array}{c}\alpha^{*}\left[(1-2\nu)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]+\lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right]\\ -\left[\alpha^{*2}\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]+2(1-\nu)\lambda\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right]\\ -\alpha^{*}\beta^{*}\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]\\ \frac{E^{*}}{E}(1+\nu)\frac{\alpha^{*}\beta^{*}}{\lambda^{2}}\left[\sinh\left[(\lambda\left(\frac{z-H}{h}\right)\right]-\lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right]\\ \frac{E^{*}}{E}(1+\nu)\frac{1}{\lambda^{2}}\left[(\alpha^{*2}-4(1-\nu)\lambda^{2})\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]-\alpha^{*2}\lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right]\\ -\frac{E^{*}}{E}(1+\nu)\alpha^{*}\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]\end{array}\right\}, \tag{5.2a}$$

$$\overline{R}^{y}(z)=\frac{1}{4\pi(1-\nu)\lambda j_{\beta}}\left\{\begin{array}{c}\beta^{*}\left[(1-2\nu)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]+\lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right]\\ -\alpha^{*}\beta^{*}\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]\\ -\left[\beta^{*2}\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]+2(1-\nu)\lambda\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right]\\ \frac{E^{*}}{E}(1+\nu)\frac{1}{\lambda^{2}}\left[(\beta^{*2}-4(1-\nu)\lambda^{2})\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]-\beta^{*2}\lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right]\\ \frac{E^{*}}{E}(1+\nu)\frac{\alpha^{*}\beta^{*}}{\lambda^{2}}\left[\sinh\left[(\lambda\left(\frac{z-H}{h}\right)\right]-\lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right]\\ -\frac{E^{*}}{E}(1+\nu)\beta^{*}\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]\end{array}\right\}, \tag{5.2b}$$

$$\overline{R}^{z}(z)=\frac{1}{4\pi(1-\nu)}\left\{\begin{array}{c}\lambda\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]-2(1-\nu)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\\ \frac{\alpha^*}{\lambda}\left[(1-2\nu)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]-\lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(-{}_{h}H\right)\right]\right]\\ \frac{\beta^*}{\lambda}\left[(1-2\nu)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]-\lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right]\\ -\frac{E^*}{E}(1+\nu)\frac{\beta^*}{\lambda}\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]\\ -\frac{E^*}{E}(1+\nu)\frac{\alpha^*}{\lambda}\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]\\ -\frac{E^*}{E}(1+\nu)\frac{1}{\lambda}\left[(4\nu-3)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]+\lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right]\end{array}\right\}, \qquad (5.2c)$$

$$\times\cosh[\lambda(\chi-\psi)]-\chi\psi\cosh[\lambda(2-\chi-\psi)]]$$
$$+\frac{\lambda^2}{4}[((4\nu-3)\chi+4(1-\nu)-\psi)\sinh[\lambda(\chi+\psi)]$$
$$+(4(1-\nu)+\chi-\psi)\sinh[\lambda(\chi-\psi)]$$
$$+((4\nu-3)\chi-\psi)\sinh[\lambda(2-\chi-\psi)]$$
$$+(\psi-\chi)\sinh[\lambda(2+\chi-\psi)]]$$
$$-(1-\nu)\frac{\lambda}{2}[-\cosh[\lambda(\chi+\psi)]+\text{csh}[\lambda(\chi-\psi)]$$
$$+\cosh[\lambda(2-\chi-\psi)]-\cosh[\lambda(2+\chi-\psi)]]\Big\}$$
$$+\{\lambda^2(\chi-\psi)\sinh[\lambda(\chi-\psi)]-\lambda 2(1-\nu)\cosh[\lambda(\chi-\psi)]\}$$
$$\times\mathscr{H}(\chi-\psi);$$

$$\sigma_{zx}^z(x,\ y,\ z)=\frac{1}{4\pi(1-\nu)h^2}\frac{x}{\sqrt{x^2+y^2}}\int_{\lambda=0}^{\lambda=\infty}f_{zx}^z(\lambda)J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda, \tag{5.5}$$

where

$$f_{zx}^z(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{\lambda^4\chi(\psi-1)\cosh[\lambda(\chi-\psi)]$$
$$-\frac{\lambda^3}{2}[(1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)]+((4\nu-3)\chi-\psi+1)$$
$$\times\sinh[\lambda(\chi-\psi)]+\chi\psi\sinh[\lambda(2-\chi-\psi)]]$$
$$-\frac{\lambda^2}{4}[((4\nu-3)\chi+2(1-2\nu)+\psi)\cosh[\lambda(\chi+\psi)]$$
$$+(2(2\nu-1)+\chi-\psi)\cosh[\lambda(\chi-\psi)]$$
$$+((3-4\nu)\chi-\psi)\cosh[\lambda(2-\chi-\psi)]$$
$$+(\psi-\chi)\cosh[\lambda(2+\chi-\psi)]]$$
$$-(2\nu-1)\frac{\lambda}{4}[\sinh[\lambda(\chi+\psi)]+\sinh[\lambda(\chi-\psi)]$$
$$+\sinh[\lambda(2-\chi-\psi)]-\sinh[\lambda(2+\chi-\psi)]]\Big\}$$
$$+\{-\lambda^2(\chi-\psi)\cosh[\lambda(\chi-\psi)]-\lambda(2\nu-1)\sinh[\lambda(\chi-\psi)]\}$$
$$\times\mathscr{H}(\chi-\psi);$$

$$\sigma_{zy}^z(x,\ y,\ z)=\frac{1}{4\pi(1-\nu)h^2}\frac{y}{\sqrt{x^2+y^2}}\int_{\lambda=0}^{\lambda=\infty}f_{zy}^z(\lambda)J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda, \tag{5.6}$$

where

$$
\begin{aligned}
f_{zy}^{z}(\lambda) = {} & \frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{\lambda^4\chi(\psi-1)\cosh[\lambda(\chi-\psi)] \\
& -\frac{\lambda^3}{2}[(1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)] + ((4\nu-3)\chi-\psi+1) \\
& \times\sinh[\lambda(\chi-\psi)] + \chi\psi\sinh[\lambda(2-\chi-\psi)]] \\
& -\frac{\lambda^2}{4}[((4\nu-3)\chi+2(1-2\nu)+\psi)\cosh[\lambda(\chi+\psi)] \\
& +(2(2\nu-1)+\chi-\psi)\cosh[\lambda(\chi-\psi)] \\
& +((3-4\nu)\chi-\psi)\cosh[\lambda(2-\chi-\psi)] \\
& +(\psi-\chi)\cosh[\lambda(2+\chi-\psi)]] \\
& -(2\nu-1)\frac{\lambda}{4}[\sinh[\lambda(\chi+\psi)] + \sinh[\lambda(\chi-\psi)] \\
& +\sinh[\lambda(2-\chi-\psi)] - \sinh[\lambda(2+\chi-\psi)]]\Big\} \\
& +\{-\lambda^2(\chi-\psi)\cosh[\lambda(\chi-\psi)] - \lambda(2\nu-1)\sinh[\lambda(\chi-\psi)]\} \\
& \times\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\sigma_{zz}^{x}(x,\ y,\ z) = \frac{1}{4\pi(1-\nu)h^2}\,\frac{-x}{\sqrt{x^2+y^2}}\int_{\lambda=0}^{\lambda=\infty} f_{zz}^{x}(\lambda)J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda, \tag{5.7}
$$

where

$$
\begin{aligned}
f_{zz}^{x}(\lambda) = {} & \frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{\lambda^4\chi(1-\psi)\cosh[\lambda(\chi-\psi)] \\
& -\frac{\lambda^3}{2}[(1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)] + ((4\nu-3)\chi-\psi+1) \\
& \times\sinh[\lambda(\chi-\psi)] + \chi\psi\sinh[\lambda(2-\chi-\psi)]] \\
& +\frac{\lambda^2}{4}[((4\nu-3)\chi+2(1-2\nu)+\psi)\cosh[\lambda(\chi+\psi)] \\
& +(2(2\nu-1)+\chi-\psi)\cosh[\lambda(\chi-\psi)] \\
& +((3-4\nu)\chi-\psi)\cosh[\lambda(2-\chi-\psi)] + (\psi-\chi) \\
& \times\cosh[\lambda(2+\chi-\psi)]] \\
& -(2\nu-1)\frac{\lambda}{4}[\sinh[\lambda(\chi+\psi)] + \sinh[\lambda(\chi-\psi)] \\
& +\sinh[\lambda(2-\chi-\psi)] - \sinh[\lambda(2+\chi-\psi)]]\Big\} \\
& +\{\lambda^2(\chi-\psi)\cosh[\lambda(\chi-\psi)] - \lambda(2\nu-1)\sinh[\lambda(\chi-\psi)]\} \\
& \times\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$\sigma_{zx}^{x}(x, y, z) = \frac{1}{4\pi(1-\nu)h^2}\left\{\int_{\lambda=0}^{\lambda=\infty} f_{zx_1}^{x}(\lambda) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda \right.$$
$$+\tfrac{1}{2}\left[\int_{\lambda=0}^{\lambda=\infty}\left(f_{zx_2}^{x}(\lambda)+f_{zx_3}^{x}(\lambda)\right) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda\right.$$
$$\left.\left.+\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\left(f_{zx_2}^{x}(\lambda)-f_{zx_3}^{x}(\lambda)\right) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda\right]\right\}, \tag{5.8}$$

where

$$f_{zx_1}^{x}(\lambda) = \frac{1}{\sinh^2(\lambda)-\lambda^2}\left\{(1-\nu)\frac{\lambda}{2}\left[\cosh[\lambda(\chi+\psi)] - \cosh[\lambda(\chi-\psi)]\right.\right.$$
$$\left.\left.-\cosh[\lambda(2-\chi-\psi)] + \cosh[\lambda(2+\chi-\psi)]\right]\right\}$$
$$+\{-\lambda 2(1-\nu)\cosh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi),$$

$$f_{zx_2}^{x}(\lambda) = \frac{1}{\sinh^2(\lambda)-\lambda^2}\left\{\lambda^4\chi(\psi-1)\sinh[\lambda(\chi-\psi)]\right.$$
$$+\frac{\lambda^3}{2}\left[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)] + ((4\nu-3)\chi+\psi-1)\right.$$
$$\left.\times\cosh[\lambda(\chi-\psi)] - \chi\psi\cosh[\lambda(2-\chi-\psi)]\right]$$
$$+\frac{\lambda^2}{4}\left[((3-4\nu)\chi+4(\nu-1)+\psi)\sinh[\lambda(\chi+\psi)]\right.$$
$$+(4(\nu-1)-\chi+\psi)\sinh[\lambda(\chi-\psi)]$$
$$+((3-4\nu)\chi+\psi)\sinh[\lambda(2-\chi-\psi)] + (\chi-\psi)$$
$$\left.\left.\times\sinh[\lambda(2+\chi-\psi)]\right]\right\}$$
$$+\{\lambda^2(\psi-\chi)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi),$$

and

$$f_{zx_3}^{x}(\lambda) = \frac{1}{\sinh^2(\lambda)-\lambda^2}\left\{-2(1-\nu)\lambda^3\frac{\sinh(\lambda\chi)\cosh[\lambda(1-\psi)]}{\sinh(\lambda)}\right\};$$

$$\sigma_{zy}^{x}(x, y, z) = \sigma_{zx}^{y}(z)$$
$$= \frac{1}{4\pi(1-\nu)h^2}\,\frac{-xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty} f_{zy}^{x}(\lambda) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda, \tag{5.9}$$

where

$$f_{zy}^{x}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{\lambda^4\chi(\psi-1)\sinh[\lambda(\chi-\psi)]$$
$$+\frac{\lambda^3}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)]+((4\nu-3)\chi+\psi-1)$$
$$\times\cosh[\lambda(\chi-\psi)]-\psi\cosh[\lambda(2-\chi-\psi)]]$$
$$+\frac{\lambda^2}{4}[((3-4\nu)\chi+4(\nu-1)+\psi)\sinh[\lambda(\chi+\psi)]$$
$$+(4(\nu-1)-\chi+\psi)\sinh[\lambda(\chi-\psi)]$$
$$+((3-4\nu)\chi+\psi)\sinh[\lambda(2-\chi-\psi)]+(\chi-\psi)$$
$$\times\sinh[\lambda(2+\chi-\psi)]]2(1-\nu)\lambda^3\frac{\sinh(\lambda\chi)\cosh[\lambda(1-\psi)]}{\sinh(\lambda)}\Big\}$$
$$+\{\lambda^2(\psi-\chi)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi),$$

$$\sigma_{zz}^{y}(x,y,z)=\frac{1}{4\pi(1-\nu)h^2}\frac{-y}{\sqrt{x^2+y^2}}\int_{\lambda=0}^{\lambda=\infty}f_{zz}^{y}(\lambda)J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)d\lambda, \tag{5.10}$$

where

$$f_{zz}^{y}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{\lambda^4\chi(1-\psi)\cosh[\lambda(\chi-\psi)]$$
$$-\frac{\lambda^3}{2}[(1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)]+((4\nu-3)\chi-\psi+1)$$
$$\times\sinh[\lambda(\chi-\psi)]+\chi\psi\sinh[\lambda(2-\chi-\psi)]]$$
$$+\frac{\lambda^2}{4}[((4\nu-3)\chi+2(1-2\nu)+\psi)\cosh[\lambda(\chi+\psi)]$$
$$+(2(2\nu-1)+\chi-\psi)\cosh[\lambda(\chi-\psi)]$$
$$+((3-4\nu)\chi-\psi)\cosh[\lambda(2-\chi-\psi)]+(\psi-\chi)$$
$$\times\cosh[\lambda(2+\chi-\psi)]]$$
$$-(2\nu-1)\frac{\lambda}{4}[\sinh[\lambda(\chi+\psi)]+\sinh[\lambda(\chi-\psi)]$$
$$+\sinh[\lambda(2-\chi-\psi)]-\sinh[\lambda(2+\chi-\psi)]]\Big\}$$
$$+\{\lambda^2(\chi-\psi)\cosh[\lambda(\chi-\psi)]-\lambda(2\nu-1)\sinh[\lambda(\chi-\psi)]\}$$
$$\times\mathscr{H}(\chi-\psi);$$

$$\sigma_{zy}^{y}(x,\ y,\ z)=\frac{1}{4\pi(1-\nu)h^2}\left\{\int_{\lambda=0}^{\lambda=\infty}f_{zy_1}^{y}(\lambda)J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right.$$
$$+\tfrac{1}{2}\left[\int_{\lambda=0}^{\lambda=\infty}\left(f_{zy_2}^{y}(\lambda)+f_{zy_3}^{y}(\lambda)\right)J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right.$$
$$\left.\left.-\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\left(f_{zy_2}^{y}(\lambda)-f_{zy_3}^{y}(\lambda)\right)J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right]\right\},\tag{5.11}$$

where

$$f_{zy_1}^{y}=\frac{1}{\sinh^2(\lambda)-\lambda^2}\left\{(1-\nu)\frac{\lambda}{2}[\cosh[\lambda(\chi+\psi)]-\cosh[\lambda(\chi-\psi)]\right.$$
$$\left.-\cosh[\lambda(2-\chi-\psi)]+\cosh[\lambda(2+\chi-\psi)]]\right\}$$
$$+\{-\lambda 2(1-\nu)\cosh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi),$$

$$f_{zy_2}^{y}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\left\{\lambda^4\chi(\psi-1)\sinh[\lambda(\chi-\psi)]\right.$$
$$+\frac{\lambda^3}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)]+((4\nu-3)\chi+\psi-1)$$
$$\times\cosh[\lambda(\chi-\psi)]-\chi\psi\cosh[\lambda(2-\chi-\psi)]]$$
$$+\frac{\lambda^2}{4}[((3-4\nu)\chi+4(\nu-1)+\psi)\sinh[\lambda(\chi+\psi)]$$
$$+(4(\nu-1)-\chi+\psi)\sinh[\lambda(\chi-\psi)]$$
$$+((3-4\nu)\chi+\psi)\sinh[\lambda(2-\chi-\psi)]+(\chi-\psi)$$
$$\left.\times\sinh[\lambda(2+\chi-\psi)]]\right\}$$
$$+\{\lambda^2(\psi-\chi)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi),$$

and

$$f_{zy_3}^{y}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\left\{-2(1-\nu)\lambda^3\frac{\sinh(\lambda\chi)\cosh[\lambda(1-\psi)]}{\sinh(\lambda)}\right\};$$

$$u_y^z(x,\ y,\ z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{y}{\sqrt{x^2+y^2}}\int_{\lambda=0}^{\lambda=\infty}f_y^z(\lambda)J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda,\tag{5.12}$$

where

$$f_y^z(\lambda) = \frac{1}{\sinh^2(\lambda) - \lambda^2}\Big\{\lambda^3\chi(\psi - 1)\sinh[\lambda(\chi - \psi)]$$
$$- \frac{\lambda^2}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)] + (4\nu - 3)(\chi + \psi - 1)$$
$$\times \cosh[\lambda(\chi - \psi)] - \chi\psi\cosh[\lambda(2 - \chi - \psi)]]$$
$$+ \frac{\lambda}{4}[(4\nu - 3)(\psi - \chi)\sinh[\lambda(\chi + \psi)]$$
$$+ (8(2\nu - 1)(\nu - 1) - \chi + \psi)\sinh[\lambda(\chi - \psi)]$$
$$+ (4\nu - 3)(\psi - \chi)\sinh[\lambda(2 - \chi - \psi)]$$
$$+ (\chi - \psi)\sinh[\lambda(2 + \chi - \psi)]]$$
$$+ (2\nu - 1)(\nu - 1)[\cosh[\lambda(\chi + \psi)] - \cosh[\lambda(2 - \chi - \psi)]]\Big\}$$
$$+ \{\lambda(\psi - \chi)\sinh[\lambda(\chi - \psi)]\}\mathscr{H}(\chi - \psi);$$

$$u_x^z(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{x}{\sqrt{x^2+y^2}}\int_{\lambda=0}^{\lambda=\infty} f_x^z(\lambda) J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) \mathrm{d}\lambda, \tag{5.13}$$

where

$$f_x^z(\lambda) = \frac{1}{\sinh^2(\lambda) - \lambda^2}\Big\{\lambda^3\chi(\psi - 1)\sinh[\lambda(\chi - \psi)]$$
$$- \frac{\lambda^2}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)] + (4\nu - 3)(\chi + \psi - 1)$$
$$\times \cosh[\lambda(\chi - \psi)] - \chi\psi\cosh[\lambda(2 - \chi - \psi)]]$$
$$+ \frac{\lambda}{4}[(4\nu - 3)(\psi - \chi)\sinh[\lambda(\chi + \psi)]$$
$$+ (8(2\nu - 1)(\nu - 1) - \chi + \psi)\sinh[\lambda(\chi - \psi)]$$
$$+ (4\nu - 3)(\psi - \chi)\sinh[\lambda(2 - \chi - \psi)]$$
$$+ (\chi - \psi)\sinh[\lambda(2 + \chi - \psi)]]$$
$$+ (2\nu - 1)(\nu - 1)[\cosh[\lambda(\chi + \psi)] - \cosh[\lambda(2 - \chi - \psi)]]\Big\}$$
$$+ \{\lambda(\psi - \chi)\sinh[\lambda(\chi - \psi)]\}\mathscr{H}(\chi - \psi);$$

$$u_z^z(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\int_{\lambda=0}^{\lambda=\infty} f_z^z(\lambda) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) \mathrm{d}\lambda, \tag{5.14}$$

where

$$
\begin{aligned}
f_z^z(\lambda) = {} & \frac{1}{\sinh^2(\lambda) - \lambda^2} \Big\{ \lambda^3 \chi (1-\psi) \cosh[\lambda(\chi - \psi)] \\
& + \frac{\lambda^2}{2} [(4\nu - 3)(1 + \chi - \psi) \sinh[\lambda(\chi - \psi)] + (1-\chi)(1-\psi) \\
& \times \sinh[\lambda(\chi + \psi)] + \chi\psi \sinh[\lambda(2 - \chi - \psi)]] \\
& + \frac{\lambda}{4} [(2(8\nu^2 - 12\nu + 5) + \chi - \psi) \cosh[\lambda(\chi - \psi)] \\
& + (3 - 4\nu)(2 - \chi - \psi) \cosh[\lambda(\chi + \psi)] \\
& + (3 - 4\nu)(\chi + \psi) \cosh[\lambda(2 - \chi - \psi)] \\
& + (\lambda - \chi) \cosh[\lambda(2 + \chi - \psi)]] \\
& + \tfrac{1}{4} [(8\nu^2 - 12\nu + 5) \sinh[\lambda(\chi + \psi)] \\
& + (8\nu^2 - 12\nu + 5) \sinh[\lambda(2 - \chi - \psi)] \\
& + (4\nu - 3) \sinh[\lambda(\chi - \psi)] - (4\nu - 3) \sinh[\lambda(2 + \chi - \psi)]] \Big\} \\
& + \{ \lambda(\chi - \psi) \cosh[\lambda(\chi - \psi)] + (4\nu - 3) \sinh[\lambda(\chi - \psi)] \} \\
& \times \mathscr{H}(\chi - \psi);
\end{aligned}
$$

$$
\begin{aligned}
u_y^x(x, y, z) &= u_x^y(z) \\
&= \frac{1+\nu}{4\pi(1-\nu)h} \frac{1}{E} \frac{-xy}{x^2 + y^2} \int_{\lambda=0}^{\lambda=\infty} f_y^x(\lambda) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h} \right) d\lambda,
\end{aligned}
\tag{5.15}
$$

where

$$
\begin{aligned}
f_y^x(\lambda) = {} & \frac{1}{\sinh^2(\lambda) - \lambda^2} \Big\{ \lambda^3 \chi (\psi - 1) \cosh[\lambda(\chi - \psi)] \\
& + \frac{\lambda^2}{2} [((2\nu - 1) + \chi\psi - \chi - \psi) \sinh[\lambda(\chi + \psi)] \\
& + ((2\nu - 1) + (3 - 4\nu)(\psi - \chi)) \sinh[\lambda(\chi - \psi)] \\
& + (2(\nu - 1) + \psi) \sinh[\lambda(2 - \chi - \psi)] \\
& + 2(\nu - 1) \sinh[\lambda(2 + \chi - \psi)]] \\
& - \frac{\lambda}{4} [(2(8\nu^2 - 12\nu + 5) + \chi - \psi) \cosh[\lambda(\chi - \psi)] \\
& + (3 - 4\nu)(2 - \chi - \psi) \cosh[\lambda(\chi + \psi)] \\
& + (3 - 4\nu)(\chi + \psi) \cosh[\lambda(2 - \chi - \psi)]
\end{aligned}
$$

$$+(\psi-\chi)\cosh[\lambda(2+\chi-\psi)]]$$
$$+\tfrac{1}{4}[(8\nu^2-8\nu+1)\sinh[\lambda(\chi+\psi)]$$
$$+(8\nu^2-8\nu+1)\sinh[\lambda(2-\chi-\psi)]$$
$$+\sinh[\lambda(\chi-\psi)]-\sinh[\lambda(2+\chi-\psi)]]$$
$$+4(1-\nu)\lambda^2\frac{\cosh^2(\lambda)\cosh(\lambda\chi)}{\sinh(\lambda)}\cosh[\lambda(1-\psi)]\Big\}$$
$$+\{-\lambda(\chi-\psi)\cosh[\lambda(\chi-\psi)]+\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);$$

$$u_x^x(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\left\{\int_{\lambda=0}^{\lambda=\infty}f_{x_3}^x(\lambda)J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right.$$
$$+\frac{1}{2}\left[\int_{\lambda=0}^{\lambda=\infty}\left(f_{x_1}^x(\lambda)+f_{x_2}^x(\lambda)\right)J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right.$$
$$\left.\left.+\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\left(f_{x_1}^x(\lambda)-f_{x_2}^x(\lambda)\right)J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right]\right\},\tag{5.16}$$

where

$$f_{x_1}^x(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{\lambda^3\chi(\psi-1)\cosh[\lambda(\chi-\psi)]$$
$$+\frac{\lambda^2}{2}[-(3-4\nu)(1+\chi-\psi)\sinh[\lambda(\chi-\psi)]$$
$$+(1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)]+\chi\psi\sinh[\lambda(2-\chi-\psi)]]$$
$$-\frac{\lambda}{4}[(2(8\nu^2-12\nu+5)+\chi-\psi)\cosh[\lambda(\chi-\psi)]$$
$$+(3-4\nu)(2-\chi-\psi)\cosh[\lambda(\chi+\psi)]$$
$$+(3-4\nu)(\chi+\psi)\cosh[\lambda(2-\chi-\psi)]$$
$$+(\psi-\chi)\cosh[\lambda(2+\chi-\psi)]]$$
$$+\tfrac{1}{4}[(8\nu^2-8\nu+1)\sinh[\lambda(\chi+\psi)]$$
$$+(8\nu^2-8\nu+1)\sinh[\lambda(2-\chi-\psi)]$$
$$+\sinh[\lambda(\chi-\psi)]-\sinh[\lambda(2+\chi-\psi)]]\Big\}$$
$$+\{-\lambda(\chi-\psi)\cosh[\lambda(\chi-\psi)]+\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi),$$

$$f_{x_2}^x(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\left[-4(1-\nu)\lambda^2\frac{\cosh(\lambda\chi)\cosh[\lambda(1-\psi)]}{\sinh(\lambda)}\right],$$

and

$$f^x_{x_3}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\{(1-\nu)[\sinh[\lambda(\chi+\psi)]-\sinh[\lambda(\chi-\psi)]$$
$$+\sinh[\lambda(2-\chi-\psi)]+\sinh[\lambda(2+\chi-\psi)]]\}$$
$$+\{-4(1-\nu)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);$$

$$u^x_z(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{-x}{\sqrt{x^2+y^2}}\int_{\lambda=0}^{\lambda=\infty}f^x_z(\lambda)J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda, \tag{5.17}$$

where

$$f^x_z(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{\lambda^3\chi(1-\psi)\sinh[\lambda(\chi-\psi)]$$
$$-\frac{\lambda^2}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)]+(3-4\nu)(1-\chi-\psi)$$
$$\times\cosh[\lambda(\chi-\psi)]-\chi\psi\cosh[\lambda(2-\chi-\psi)]]$$
$$-\frac{\lambda}{4}[(3-4\nu)(\chi-\psi)\sinh[\lambda(\chi+\psi)]$$
$$+(8(2\nu-1)(\nu-1)-\chi+\psi)\sinh[\lambda(\chi-\psi)]$$
$$+(3-4\nu)(\chi-\psi)\sinh[\lambda(2-\chi-\psi)]$$
$$+(\chi-\psi)\sinh[\lambda(2+\chi-\psi)]]$$
$$+(2\nu-1)(\nu-1)[\cosh[\lambda(\chi+\psi)]-\cosh[\lambda(2-\chi-\psi)]]\Big\}$$
$$+\{-\lambda(\psi-\chi)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);$$

$$u^y_y(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\left\{\int_{\lambda=0}^{\lambda=\infty}f^y_{y_3}(\lambda)J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right.$$
$$+\tfrac{1}{2}\left[\int_{\lambda=0}^{\lambda=\infty}\left(f^y_{y_1}(\lambda)+f^y_{y_2}(\lambda)\right)J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right.$$
$$\left.\left.-\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\left(f^y_{y_1}(\lambda)-f^y_{y_2}(\lambda)\right)J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right]\right\}, \tag{5.18}$$

where

$$f^y_{y_1}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{\lambda^3\chi(\psi-1)\cosh[\lambda(\chi-\psi)]$$
$$+\frac{\lambda^2}{2}[-(3-4\nu)(1+\chi-\psi)\sinh[\lambda(\chi-\psi)]$$

$$+(1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)]+\chi\psi\sinh[\lambda(2-\chi-\psi)]]$$
$$-\frac{\lambda}{4}[(2(8\nu^2-12\nu+5)+\chi-\psi)\cosh[\lambda(\chi-\psi)]$$
$$+(3-4\nu)(2-\chi-\psi)\cosh[\lambda(\chi+\psi)]$$
$$+(3-4\nu)(\chi+\psi)\cosh[\lambda(2-\chi-\psi)]$$
$$+(\psi-\chi)\cosh[\lambda(2+\chi-\psi)]]$$
$$+\tfrac{1}{4}[(8\nu^2-8\nu+1)\sinh[\lambda(\chi+\psi)]$$
$$+(8\nu^2-8\nu+1)\sinh[\lambda(2-\chi-\psi)]$$
$$+\sinh[\lambda(\chi-\psi)]-\sinh[\lambda(2+\chi-\psi)]]\Big\}$$
$$+\{-\lambda(\chi-\psi)\cosh[\lambda(\chi-\psi)]+\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi),$$
$$f^y_{y_2}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\left[-4(1-\nu)\lambda^2\frac{\cosh(\lambda\chi)\cosh[\lambda(1-\psi)]}{\sinh(\lambda)}\right],$$

and

$$f^y_{y_3}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\{(1-\nu)[\sinh[\lambda(\chi+\psi)]-\sinh[\lambda(\chi-\psi)]$$
$$+\sinh[\lambda(2-\chi-\psi)]+\sinh[\lambda(2+\chi-\psi)]]\}$$
$$+\{-4(1-\nu)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);$$
$$u^y_z(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{-y}{\sqrt{x^2+y^2}}\int_{\lambda=0}^{\lambda=\infty}f^y_z(\lambda)J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda,\tag{5.19}$$

where

$$f^y_z(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{\lambda^3\chi(1-\psi)\sinh[\lambda(\chi-\psi)]$$
$$-\frac{\lambda^2}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)]+(3-4\nu)(1-\chi-\psi)$$
$$\times\cosh[\lambda(\chi-\psi)]-\chi\psi\cosh[\lambda(2-\chi-\psi)]]$$
$$-\frac{\lambda}{4}[(3-4\nu)(\chi-\psi)\sinh[\lambda(\chi+\psi)]$$
$$+(8(2\nu-1)(\nu-1)-\chi+\psi)\sinh[\lambda(\chi-\psi)]$$
$$+(3-4\nu)(\chi-\psi)\sinh[\lambda(2-\chi-\psi)]$$
$$+(\chi-\psi)\sinh[\lambda(2+\chi-\psi)]]$$
$$+(2\nu-1)(\nu-1)[\cosh[\lambda(\chi+\psi)]-\cosh[\lambda(2-\chi-\psi)]]\Big\}$$
$$+\{-\lambda(\psi-\chi)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi).$$

Making use of the transformed stresses and displacements given by (5.3) and by means of the equation (3.7), after performing the inverse transforms in connection with (3.1) and by using Appendix A, we obtain:

$$\sigma_{xx}^{z}(x, y, z) = \frac{\nu}{1-\nu}\sigma_{zz}^{z}(x, y, z) + \frac{1}{8\pi(1-\nu)^2 h^2} \times \left\{(1+\nu)\int_{\lambda=0}^{\lambda=\infty} \lambda f_x^z(\lambda) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda + (1-\nu)\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty} \lambda f_x^z(\lambda) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda\right\}, \tag{5.20}$$

$$\sigma_{yy}^{z}(x, y, z) = \frac{\nu}{1-\nu}\sigma_{zz}^{z}(x, y, z) + \frac{1}{8\pi(1-\nu)^2 h^2} \times \left\{(1+\nu)\int_{\lambda=0}^{\lambda=\infty} \lambda f_y^z(\lambda) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda + (1-\nu)\frac{x^2-y^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty} \lambda f_y^z(\lambda) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda\right\}, \tag{5.21}$$

$$\sigma_{xy}^{z}(x, y, z) = -\frac{1}{4\pi(1-\nu)h^2}\frac{xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty} \lambda f_y^z(\lambda) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda, \tag{5.22}$$

$$\sigma_{xx}^{x}(x, y, z) = \frac{\nu}{1-\nu}\sigma_{zz}^{x}(x, y, z) + \frac{1}{16\pi(1-\nu)^2 h^2}\frac{x}{\sqrt{x^2+y^2}} \times \left\{(1-\nu)\frac{x^2-3y^2}{x^2+y^2}\cdot\int_{\lambda=0}^{\lambda=\infty} \lambda\left[f_{x1}^x(\lambda) - f_{x_2}^x(\lambda)\right] J_3\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda - \int_{\lambda=0}^{\lambda=\infty} \lambda\left[(3+\nu)f_{x_1}^x(\lambda) + (1-\nu)f_{x_2}^x(\lambda) + 4f_{x_3}^x(\lambda)\right] \times J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda\right\} \tag{5.23}$$

$$\sigma_{yy}^{x}(x, y, z) = \frac{\nu}{1-\nu}\sigma_{zz}^{x}(x, y, z) + \frac{1}{16\pi(1-\nu)^2 h^2}\frac{x}{\sqrt{x^2+y^2}} \times \left\{(\nu-1)\frac{x^2-3y^2}{x^2+y^2}\cdot\int_{\lambda=0}^{\lambda=\infty} \lambda\left[f_{x_1}^x(\lambda) - f_{x_2}^x(\lambda)\right] J_3\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda\right.$$

$$-\int_{\lambda=0}^{\lambda=\infty}\lambda\left[(1+3\nu)f_{x_1}^{x}(\lambda)-(1-\nu)f_{x_2}^{x}(\lambda)+4\nu f_{x_3}^{x}(\lambda)\right]$$

$$\left.\times J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\,\mathrm{d}\lambda\right\}, \tag{5.24}$$

$$\sigma_{xy}^{x}(x,\,y,\,z)=-\frac{1}{16\pi(1-\nu)h^2}\frac{y}{\sqrt{x^2+y^2}}$$

$$\times\left\{\frac{y^2-3x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{x1}^{x}(\lambda)-f_{x2}^{x}(\lambda)\right]J_3\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}y\right.$$

$$\left.+\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{x1}^{x}(\lambda)+f_{x2}^{x}(\lambda)+2f_{x3}^{x}(\lambda)\right]J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right\}, \tag{5.25}$$

$$\sigma_{xx}^{y}(x,\,y,\,z)=\frac{\nu}{1-\nu}\sigma_{zz}^{y}(x,\,y,\,z)+\frac{1}{16\pi(1-\nu)^2h^2}\frac{y}{\sqrt{x^2+y^2}}$$

$$\times\left\{(\nu-1)\frac{y^2-3x^2}{x^2+y^2}\cdot\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{y1}^{y}(\lambda)-f_{y2}^{y}(\lambda)\right]J_3\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right.$$

$$-\int_{\lambda=0}^{\lambda=\infty}\lambda\left[(1+3\nu)f_{y1}^{y}(\lambda)-(1-\nu)f_{y2}^{y}(\lambda)+4\nu f_{y3}^{y}(\lambda)\right]$$

$$\left.\times J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\,\mathrm{d}\lambda\right\}, \tag{5.26}$$

$$\sigma_{yy}^{y}(x,\,y,\,z)=\frac{\nu}{1-\nu}\sigma_{zz}^{y}(x,\,y,\,z)+\frac{1}{16\pi(1-\nu)^2h^2}\frac{y}{\sqrt{x^2+y^2}}$$

$$\times\left\{(1-\nu)\frac{y^2-3x^2}{x^2+y^2}\cdot\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{y1}^{y}(\lambda)-f_{y2}^{y}(\lambda)\right]J_3\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right.$$

$$-\int_{\lambda=0}^{\lambda=\infty}\lambda\left[(3+\nu)f_{y1}^{y}(\lambda)+(1-\nu)f_{y2}^{y}(\lambda)+4f_{y3}^{y}(\lambda)\right]$$

$$\left.\times J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\,\mathrm{d}\lambda\right\}, \tag{5.27}$$

$$\sigma_{xy}^{y}(x,\,y,\,z)=-\frac{1}{16\pi(1-\nu)h^2}\frac{x}{\sqrt{x^2+y^2}}$$

$$\times\left\{\frac{x^2-3y^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{y1}^{y}(\lambda)-f_{y2}^{y}(\lambda)\right]J_3\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right.$$

$$\left.+\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{y1}^{y}(\lambda)+f_{y2}^{y}(\lambda)+2f_{y3}^{y}(\lambda)\right]J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\right\}, \tag{5.28}$$

where the $f(\lambda)$ are functions of λ, χ and ψ, and they are defined in (5.4) to (5.19).

The analytical expressions obtained above are given in terms of integrals. Close inspection of the expressions for the stresses σ_{zz}, σ_{zx}, σ_{zy}, σ_{xy} (when the unit load is applied along the x, y or z directions), or σ_{xx} and σ_{yy} (when the unit load is applied along the x or y directions), see equations (5.4) to (5.11) and (5.22) to (5.28), demonstrates that the resulting integrands exist and are well behaved for every $\lambda \in [0, \infty)$. For $\lambda \to 0$, this was shown by expanding the integrands in ascending powers of λ and proving that the resulting expressions vanish as $\lambda \to 0$. For $\lambda \to \infty$, this was shown by replacing the hyperbolic functions involved by their equivalent exponential forms and demonstrating that the limit of the resulting expression, as $\lambda \to \infty$, vanishes.

On the other hand, analysis of the equivalent expressions for the displacements, (5.12) to (5.19), and for the stresses σ_{xx} and σ_{yy} (when the unit load is applied along the z direction), (5.20) and (5.21), showed that although the integrands involved were well behaved as $\lambda \to \infty$, they became singular as $\lambda \to 0$. In fact, expansion of these integrands in ascending powers of λ revealed terms of the form

$$A\left(\chi, \psi, \frac{R}{h}\right)\lambda^{-3} + B\left(\chi, \psi, \frac{R}{h}\right)\lambda^{-1}, \quad \lambda \to 0 \tag{5.29}$$

where A and B were known functions of $\chi = \dfrac{z}{h}$, $\psi = \dfrac{H}{h}$ and $\dfrac{R}{h} = \dfrac{\sqrt{x^2 + y^2}}{h}$ respectively.

The singular behavior of the integrands in (5.12) to (5.21) indicates that the resulting expressions for the displacements are non-convergent and that the above solution should be critically reexamined.

c. *Proposed modifications*

The construction of the final solution to our problem was suggested by the observation that simple subtraction of terms of the form

$$A\left(\chi, \psi, \frac{R}{h}\right)\lambda^{-3} + B\left(\chi, \psi, \frac{R}{h}\right)\lambda^{-1}e^{-\lambda} \tag{5.30}$$

from the original integrands resulted in integrals, for the displacements and stresses (σ^z_{xx} and σ^z_{yy}), which were convergent. It should be noted here that expression (5.30) reduces to (5.29) as $\lambda \to 0$. The inclusion of the multiplying factor $e^{-\lambda}$ in the λ^{-1} term of (5.29) ensures the integrability of the final expressions for the displacements and stresses.

It was further observed that the functions $A\lambda^{-3} + B\lambda^{-1}e^{-\lambda}$ of the transform variable $\lambda = \sqrt{\alpha^{*2} + \beta^{*2}}$ represent Fourier transforms of displacements contributing nothing to the transformed stresses $\bar{\bar{\sigma}}_{zz}$, $\bar{\bar{\sigma}}_{zx}$, $\bar{\bar{\sigma}}_{zy}$, thus automatically satisfying the zero traction boundary conditions at $z = h$ and $z = 0$. This was also consistent with the fact that the kernels of the integral expressions (5.4) to (5.11) for the σ_{zz}, σ_{zx}, σ_{zy} stresses do not involve singular terms as $\lambda \to 0$.

In addition it was shown that the stress-displacement state vector $(\bar{\sigma}_{zz}, \bar{\sigma}_{zx}, \bar{\sigma}_{zy}, \bar{u}_y, \bar{u}_x, \bar{u}_z)^T$ corresponding to the transformed displacements $A\lambda^{-3} + B\lambda^{-1}e^{-\lambda}$ also provides solutions of the transformed governing equations (3.3).

Motivated by the above observations, we propose here a solution constructed by simply subtracting singular functions of the form (5.30) from the integrands of the displacements (5.12) to (5.19). As mentioned above, the resulting displacements are convergent and give rise to stresses σ_{zz}, σ_{zx}, σ_{zy}, which are identical to the ones in equations (5.4) to (5.11).

In the next section we will present all displacement and stress components resulting from the modified solution. We shall then formally prove that the proposed fields satisfy all field equations, boundary conditions and reduce to the well known solution for a point load in an infinite domain (Kelvin state), as the point of application of the load in approached.

6. The proposed solution

$$\sigma^z_{zz}(x, y, z) = \frac{1}{4p(1-\nu)h^2} \int_{\lambda=0}^{\lambda=\infty} f^z_{zz}(\lambda) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda, \tag{6.1}$$

$$\sigma^z_{zx}(x, y, z) = \frac{1}{4\pi(1-\nu)h^2} \frac{x}{\sqrt{x^2+y^2}} \int_{\lambda=0}^{\lambda=\infty} f^z_{zx}(\lambda) J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda, \tag{6.2}$$

$$\sigma^z_{zy}(x, y, z) = \frac{1}{4\pi(1-\nu)h^2} \frac{y}{\sqrt{x^2+y^2}} \int_{\lambda=0}^{\lambda=\infty} f^z_{zy}(\lambda) J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda, \tag{6.3}$$

$$\sigma^x_{zz}(x, y, z) = \frac{1}{4\pi(1-\nu)h^2} \frac{-x}{\sqrt{x^2+y^2}} \int_{\lambda=0}^{\lambda=\infty} f^z_{zz}(\lambda) J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda, \tag{6.4}$$

$$\begin{aligned}\sigma^x_{zx}(x, y, z) = \frac{1}{4\pi(1-\nu)h^2} \Bigg\{ &\int_{\lambda=0}^{\lambda=\infty} f^x_{zx1}(\lambda) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda \\ &+ \tfrac{1}{2}\Bigg[\int_{\lambda=0}^{\lambda=\infty} \left(f^x_{zx_2}(\lambda) + f^x_{zx_3}(\lambda)\right) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda \\ &+ \frac{y^2-x^2}{x^2+y^2} \int_{\lambda=0}^{\lambda=\infty} \left(f^x_{zx_2}(\lambda) - f^x_{zx_3}(\lambda)\right) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda \Bigg]\Bigg\},\end{aligned} \tag{6.5}$$

$$\sigma_{zy}^{x}(x, y, z) = \frac{1}{4\pi(1-\nu)h^2} \frac{-xy}{x^2+y^2} \int_{\lambda=0}^{\lambda=\infty} f_{zy}^{x}(\lambda) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) \mathrm{d}\lambda, \tag{6.6}$$

$$\sigma_{zz}^{y}(x, y, z) = \frac{1}{4\pi(1-\nu)h^2} \frac{-y}{\sqrt{x^2+y^2}} \int_{\lambda=0}^{\lambda=\infty} f_{zz}^{y}(\lambda) J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) \mathrm{d}\lambda, \tag{6.7}$$

$$\sigma_{zx}^{y}(x, y, z) = \frac{1}{4\pi(1-\nu)h^2} \frac{-xy}{x^2+y^2} \int_{\lambda=0}^{\lambda=\infty} f_{zy}^{x}(\lambda) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) \mathrm{d}\lambda, \tag{6.8}$$

$$\begin{aligned}\sigma_{zy}^{y}(x, y, z) = \frac{1}{4\pi(1-\nu)h^2} \Bigg\{ & \int_{\lambda=0}^{\lambda=\infty} f_{zy_1}^{y}(\lambda) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) \mathrm{d}\lambda \\ & + \tfrac{1}{2}\Bigg[\int_{\lambda=0}^{\lambda=\infty} \left(f_{zy_2}^{y}(\lambda) + f_{zy_3}^{y}(\lambda)\right) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) \mathrm{d}\lambda \\ & - \frac{y^2-x^2}{x^2+y^2} \int_{\lambda=0}^{\lambda=\infty} \left(f_{zy_2}^{y}(\lambda) - f_{zy_3}^{y}(\lambda)\right) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) \mathrm{d}\lambda\Bigg]\Bigg\},\end{aligned} \tag{6.9}$$

$$\begin{aligned}\sigma_{xx}^{z}(x, y, z) = {} & \frac{\nu}{1-\nu}\sigma_{zx}^{z}(x, y, z) + \frac{1}{8\pi(1-\nu)^2 h^2} \\ & \times \Bigg\{(1+\nu)\int_{\lambda=0}^{\lambda=\infty} \lambda\Bigg[f_x^{z}(\lambda) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) \\ & - 12(1-\nu)^2(2\chi-1)\frac{e^{-\lambda}}{\lambda}\Bigg] \mathrm{d}\lambda \\ & + (1-\nu)\frac{y^2-x^2}{x^2+y^2} \int_{\lambda=0}^{\lambda=\infty} \lambda f_x^{z}(\lambda) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) \mathrm{d}\lambda\Bigg\},\end{aligned} \tag{6.10}$$

$$\begin{aligned}\sigma_{yy}^{z}(x, y, z) = {} & \frac{\nu}{1-\nu}\sigma_{zz}^{z}(x, y, z) + \frac{1}{8\pi(1-\nu)^2 h^2} \\ & \times \Bigg\{(1+\nu)\int_{\lambda=0}^{\lambda=\infty} \lambda\Bigg[f_y^{z}(\lambda) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) \\ & - 12(1-\nu)^2(2\chi-1)\frac{e^{-\lambda}}{\lambda}\Bigg] \mathrm{d}\lambda\end{aligned}$$

$$+ (1-\nu)\frac{x^2-y^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda f_y^z(\lambda)J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\Bigg\}, \tag{6.11}$$

$$\sigma_{xy}^z(x,\ y,\ z) = -\frac{1}{4\pi(1-\nu)h^2}\frac{xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda f_y^z(\lambda)J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda, \tag{6.12}$$

$$\sigma_{xx}^x(x,\ y,\ z) = \frac{\nu}{1-\nu}\sigma_{zz}^x(x,\ y,\ z) + \frac{1}{16\pi(1-\nu)^2h^2}\frac{x}{\sqrt{x^2+y^2}}$$
$$\times\Bigg\{(1-\nu)\frac{x^2-3y^2}{x^2+y^2}\cdot\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{x1}^x(\lambda)-f_{x2}^x(\lambda)\right]J_3\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda$$
$$-\int_{\lambda=0}^{\lambda=\infty}\lambda\left[(3+\nu)f_{x1}^x(\lambda)+(1-\nu)f_{x2}^x(\lambda)+4f_{x3}^x(\lambda)\right]$$
$$\times J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\Bigg\}, \tag{6.13}$$

$$\sigma_{yy}^x(x,\ y,\ z) = \frac{\nu}{1-\nu}\sigma_{zz}^x(x,\ y,\ z) + \frac{1}{16\pi(1-\nu)^2h^2}\frac{x}{\sqrt{x^2+y^2}}$$
$$\times\Bigg\{(\nu-1)\frac{x^2-3y^2}{x^2+y^2}\cdot\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{x1}^x(\lambda)-f_{x2}^x(\lambda)\right]J_3\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda$$
$$-\int_{\lambda=0}^{\lambda=\infty}\lambda\left[(1+3\nu)f_{x1}^x(\lambda)-(1-\nu)f_{x2}^x(\lambda)+4\nu f_{x3}^x(\lambda)\right]$$
$$\times J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\Bigg\}, \tag{6.14}$$

$$\sigma_{xy}^x(x,\ y,\ z) = -\frac{1}{16\pi(1-\nu)h^2}\frac{y}{\sqrt{x^2+y^2}}$$
$$\times\Bigg\{\frac{y^2-3x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{x1}^x(\lambda)-f_{x2}^x(\lambda)\right]J_3\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda$$
$$+\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{x1}^x(\lambda)+f_{x2}^x(\lambda)+2f_{x3}^x(\lambda)\right]J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\Bigg\}, \tag{6.15}$$

$$\sigma_{xx}^y(x,\ y,\ z) = \frac{\nu}{1-\nu}\sigma_{zz}^y(x,\ y,\ z) + \frac{1}{16\pi(1-\nu)^2h^2}\frac{y}{\sqrt{x^2+y^2}}$$
$$\times\Bigg\{(\nu-1)\frac{y^2-3x^2}{x^2+y^2}\cdot\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{y1}^y(\lambda)-f_{y2}^y(\lambda)\right]J_3\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda$$

$$-\int_{\lambda=0}^{\lambda=\infty}\lambda\left[(1+3\nu)f_{y1}^{y}(\lambda)-(1-\nu)f_{y2}^{y}(\lambda)+4\nu f_{y3}^{y}(\lambda)\right]$$

$$\times J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\,\mathrm{d}\lambda\Bigg\}, \tag{6.16}$$

$$\sigma_{yy}^{y}(x,\ y,\ z)=\frac{\nu}{1-\nu}\sigma_{zz}^{y}(x,\ y,\ z)+\frac{1}{16\pi(1-\nu)^2h^2}\frac{y}{\sqrt{x^2+y^2}}$$

$$\times\Bigg\{(1-\nu)\frac{y^2-3x^2}{x^2+y^2}\cdot\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{y1}^{y}(\lambda)-f_{y2}^{y}(\lambda)\right]J_3\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda$$

$$-\int_{\lambda=0}^{\lambda=\infty}\lambda\left[(3+\nu)f_{y1}^{y}(\lambda)+(1-\nu)f_{y2}^{y}(\lambda)+4f_{y3}^{y}(\lambda)\right]$$

$$\times J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\,\mathrm{d}\lambda\Bigg\}, \tag{6.17}$$

$$\sigma_{xy}^{y}(x,\ y,\ z)=-\frac{1}{16\pi(1-\nu)h^2}\frac{x}{\sqrt{x^2+y^2}}$$

$$\times\Bigg\{\frac{x^2-3y^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{y1}^{y}(\lambda)-f_{y2}^{y}(\lambda)\right]J_3\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda$$

$$+\int_{\lambda=0}^{\lambda=\infty}\lambda\left[f_{y1}^{y}(\lambda)+f_{y2}^{y}(\lambda)+2f_{y3}^{y}(\lambda)\right]J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda\Bigg\}, \tag{6.18}$$

$$u_z^z(x,\ y,\ z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\int_{\lambda=0}^{\lambda=\infty}$$

$$\times\Bigg\{f_z^z(\lambda)J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)-\frac{24(1-\nu)^2}{\lambda^3}$$

$$-\Bigg[12\nu(1-\nu)(\chi+\psi-\chi^2-\psi^2)+\tfrac{24}{5}(1-\nu)^2$$

$$-6(1-\nu)^2\frac{x^2+y^2}{h^2}\Bigg]\frac{e^{-\lambda}}{\lambda}\Bigg\}\,\mathrm{d}\lambda, \tag{6.19}$$

$$u_x^z(x,\ y,\ z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{x}{\sqrt{x^2+y^2}}\int_{\lambda=0}^{\lambda=\infty}\Bigg\{f_x^z(\lambda)J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)$$

$$-6(1-\nu)^2\frac{\sqrt{x^2+y^2}}{h}(2\chi-1)\frac{e^{-\lambda}}{\lambda}\Bigg\}\,\mathrm{d}\lambda, \tag{6.20}$$

$$u_y^z(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{y}{\sqrt{x^2+y^2}}\int_{\lambda=0}^{\lambda=\infty}\left\{ f_y^z(\lambda) J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\right.$$

$$\left. - 6(1-\nu)^2\frac{\sqrt{x^2+y^2}}{h}(2\chi-1)\frac{e^{-\lambda}}{\lambda}\right\} d\lambda, \qquad (6.21)$$

$$u_z^x(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{-x}{\sqrt{x^2+y^2}}\int_{\lambda=0}^{\lambda=\infty}\left\{ f_z^x(\lambda) J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\right.$$

$$\left. - 6(1-\nu)^2(2\psi-1)\frac{e^{-\lambda}}{\lambda}\right\} d\lambda, \qquad (6.22)$$

$$u_x^x(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}$$

$$\times\left\{\int_{\lambda=0}^{\lambda=\infty}\left\{\left[f_{x_3}^x(\lambda) + \tfrac{1}{2}\left(f_{x_1}^x(\lambda) + f_{x_1}^x(\lambda)\right)\right] J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\right.\right.$$

$$\left. -(\nu-1)[6(1-\nu)(\chi+\psi-2\chi\psi) + 2(2\nu-3)]\frac{e^{-\lambda}}{\lambda}\right\} d\lambda$$

$$\left. + \frac{1}{2}\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\left(f_{x_1}^x(\lambda) - f_{x_2}^x(\lambda)\right) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda\right\}, \qquad (6.23)$$

$$u_y^x(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{-xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty} f_y^x(\lambda) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda, \qquad (6.24)$$

$$u_z^y(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{-y}{\sqrt{x^2+y^2}}$$

$$\times\int_{\lambda=0}^{\lambda=\infty}\left\{ f_z^y(\lambda) J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) 6(1-\nu)^2(2\psi-1)\frac{e^{-\lambda}}{\lambda}\right\} d\lambda, \qquad (6.25)$$

$$u_x^y(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{-xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty} f_y^x(\lambda) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) d\lambda, \qquad (6.26)$$

$$u_y^y(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)h} \frac{1}{E}$$
$$\times\left\{\int_{\lambda=0}^{\lambda=\infty}\left\{\left[f_{y_3}^y(\lambda) + \tfrac{1}{2}\left(f_{y_1}^y(\lambda) + f_{y_2}^y(\lambda)\right)\right] J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\right.\right.$$
$$\left. - (\nu-1)[6(1-\nu)(\chi+\psi-2\chi\psi) + 2(2\nu-3)]\frac{e^{-\lambda}}{\lambda}\right\} \mathrm{d}\lambda$$
$$\left. - \frac{1}{2}\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\left(f_{y_1}^y(\lambda) - f_{y_2}^y(\lambda)\right) J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right) \mathrm{d}\lambda\right\}. \tag{6.27}$$

Here the superscipt (x, y, z) indicates the direction of the unit load, $\chi = \frac{z}{h}$ and $\psi = \frac{H}{h}$. Also note that the $f(\lambda)$, defined in (5.4) to (5.19), are functions of λ, χ and ψ.

a. Basic features of the proposed solution

In this section we will discuss the characteristic features of the solution presented above, see expressions (6.1) to (6.27). In particular the following properties are demonstrated:

i. The expressions for the displacement field, equations (6.19) to (6.27), satisfy the displacement equations of equilibrium:

$$\frac{E}{2(1+\nu)(1-2\nu)} u_{j,ij}(x) + \frac{E}{2(1+\nu)} u_{i,jj}(x) = 0$$
$$\begin{cases} \xi = He_z, \\ \forall x \neq \xi, \\ i, j = x, y, z. \end{cases} \tag{6.28}$$

This can be verified by direct differentiation of the convergent integral expression for the displacements and substitution into (6.28).

ii. The proposed stress field satisfies the boundary conditions prescribed on the plate surfaces:

$$\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0 \quad \text{for } \frac{z}{h} = \chi = 0 \quad \text{and } \frac{z}{h} = \chi = 1.$$

This can be easily seen by inspection of equations (6.1) to (6.9).

iii. The integral of the tractions over the boundary ∂C of a cylinder of arbitrary radius ρ, $\rho > 0$ is equal to minus the point load applied at $\xi = He_z$. Letting

$$C = \{(x, y, z) \mid x^2 + y^2 \leqslant \rho^2, 0 \leqslant z \leqslant h\}$$

be the cylinder and

$$B_\rho = \{(x, y, z) \mid x^2 + y^2 = \rho^2, 0 < z < h\}$$

be its cylindrical surface, we see that the above statement is equivalent to

$$\int_{\partial C} \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}A = \int_{B_\rho} \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}A = -\boldsymbol{F}, \tag{6.29}$$

since $\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0$ on the surfaces $\chi = 0$, $\chi = 1$ of the cylinder, and $\boldsymbol{F}$ being the applied point load.

We outline the proof of the above statement in connection with a unit load applied in the positive z direction. For this case and since $\boldsymbol{n} = \boldsymbol{e}_r = \boldsymbol{e}_x \cos\theta + \boldsymbol{e}_y \sin\theta$ on the cylindrical surface, the z component of the left hand side of (6.29) becomes

$$\rho \int_0^{2\pi} \int_0^h \left[\sigma_{zx}^z \mid_{R=\rho} \cos\theta + \sigma_{zy}^z \mid_{R=\rho} \sin\theta \right] \mathrm{d}\theta \, \mathrm{d}z. \tag{6.30}$$

Through use of (6.2), (6.30) can be written as

$$\frac{\rho}{2(1-\nu)h} \int_0^\infty \left[J_1\left(\frac{\lambda\rho}{h}\right) \int_0^1 f_{zx}^z(\lambda, \chi, \psi) \, \mathrm{d}\chi \right] \mathrm{d}\lambda, \tag{6.31}$$

where $\chi = \dfrac{z}{h}$, $\dfrac{H}{h} = \psi$.

Substitution for $f_{zx}^z(\lambda, \chi, \psi)$ from (5.5) yields $\int_0^1 f_{zx}^z(\lambda, \chi, \psi) \, \mathrm{d}\chi = -2(1 - \nu)$ and (6.31) becomes

$$-\frac{\rho}{h} \int_0^\infty J_1\left(\frac{\lambda\rho}{h}\right) \mathrm{d}\lambda = -1,$$

consistent with the requirements of equation (6.29).

Proofs of the validity of (6.29) for unit loads applied along the x and y directions are entirely analogous to the above.

iv. The proposed stress and displacement fields have the property:

$$\left.\begin{aligned} \boldsymbol{\sigma}(x) &= \mathrm{o}(r^{-2}) \\ \boldsymbol{u}(x) &= \mathrm{o}(r^{-1}) \end{aligned}\right\} r \to 0, \quad r = \left(x^2 + y^2 + (z - H)^2\right)^{1/2} > 0.$$

In particular the displacements and stresses of the present solution reduce to the equivalent ones predicted by Kelvin's solution, as the point of application of the load is approached.

We will outline the proof of the above by making use of a specific stress component corresponding to a unit load in the positive z direction. The complete proof for all displacement and stress components for concentrated loads along any direction is entirely analogous.

We consider the stress component σ^z_{zx} given in (6.2). As was stated at the end of Section 5, the integrand $f^z_{zx}(\lambda, \chi, \psi)J_1(\lambda \frac{R}{h})$ of (6.2) is well behaved for every $\lambda \in [0, \infty)$ and in particular

$$\lim_{\lambda \to 0} \left[f^z_{zx}(\lambda, \chi, \psi) J_1\left(\lambda \frac{R}{h}\right)\right] = 0.$$

This allows us to replace the integral of (6.2) by its Cauchy principal value:

$$\sigma^z_{zx} = \frac{1}{4\pi(1-\nu)h^2} \frac{x}{R} \int_{\lambda=0^+}^{\infty} f^z_{zx}(\lambda, \chi, \psi) J_1\left(\lambda \frac{R}{h}\right) d\lambda, \tag{6.32}$$

where $R = (x^2 + y^2)^{1/2} > 0$, $\chi = \frac{z}{h}$, $\psi = \frac{H}{h}$ and $f^z_{zx}(\lambda, \chi, \psi)$ is given in (5.5).

We now choose to replace the hyperbolic functions in (5.5) by their equivalent exponential forms. By doing so $f^z_{zx}(\lambda, \chi, \psi)$ can be represented as

$$\begin{aligned}
f^z_{zx} = \frac{1}{1 + [e^{-4\lambda} - (2 + 4\lambda^2)\, e^{-2\lambda}]} \Bigg\{ & 2\lambda^4 \chi(\psi - 1)(e^{-\lambda(2-\chi+\psi)} + e^{-\lambda(2+\chi-\psi)}) \\
& -\lambda^3 [(1-\chi)(1-\psi)(e^{-\lambda(2-\chi-\psi)} - e^{-\lambda(2+\chi+\psi)}) + ((4\nu - 3)\chi - \psi + 1) \\
& \times (e^{-\lambda(2-\chi+\psi)} - e^{-\lambda(2+\chi-\psi)}) + \chi\psi(e^{-\lambda(\chi+\psi)} - e^{-\lambda(4-\chi-\psi)})] \\
& - \frac{\lambda^2}{2}[((4\nu - 3)\chi + 2(1-2\nu) + \psi)(e^{-\lambda(2-\chi-\psi)} + e^{-\lambda(2+\chi+\psi)}) \\
& + (\psi - \chi)(e^{\lambda(\chi-\psi)} + e^{-\lambda(4+\chi-\psi)}) \\
& + (2(2\nu - 1) + \chi - \psi)(e^{-\lambda(2-\chi+\psi)} + e^{-\lambda(2+\chi+\psi)}) \\
& + ((3 - 4\nu)\chi - \psi)(e^{-\lambda(\chi+\psi)} + e^{-\lambda(4-\chi-\psi)})] \\
& - (2\nu - 1)\frac{\lambda}{2}[e^{-\lambda(2-\chi-\psi)} - e^{-\lambda(2+\chi+\psi)} + e^{-\lambda(2-\chi+\psi)} - e^{-\lambda(2+\chi-\psi)} \\
& + e^{-\lambda(\chi+\psi)} - e^{-\lambda(4-\chi-\psi)} - e^{\lambda(\chi-\psi)} + e^{-\lambda(4+\chi-\psi)}] \\
& + \left\langle -(\chi - \psi)\frac{\lambda^2}{2} \right\rangle [e^{\lambda(\chi-\psi)} + e^{-\lambda(\chi-\psi)} + e^{-\lambda(4-\chi+\psi)} + e^{-\lambda(4+\chi-\psi)} \\
& - 2\, e^{-\lambda(2-\chi+\psi)} - 2\, e^{-\lambda(2+\chi+\psi)}] \\
& + 2(\chi - \psi)\lambda^3 [e^{-\lambda(2-\chi+\psi)} + e^{-\lambda(2+\chi-\psi)}] \\
& - \frac{(2\nu - 1)}{2}\lambda [e^{\lambda(\chi-\psi)} - e^{-\lambda(\chi-\psi)} + e^{-\lambda(4-\chi-\psi)} - e^{-\lambda(4+\chi-\psi)} \\
& - 2\, e^{-\lambda(2-\chi+\psi)} + 2\, e^{-\lambda(2+\chi-\psi)}] \\
& + \left\langle 2(2\nu - 1)\lambda^3 [e^{-\lambda(2-\chi+\psi)} - e^{-\lambda(2+\chi-\psi)}] \right\rangle \mathscr{H}(\chi - \psi) \Bigg\}.
\end{aligned} \tag{6.33}$$

We first observe that (6.33) involves only negative exponentials in λ (the positive ones, which exist for $(\chi-\psi)>0$, cancel out).

Also the factor $(1+[e^{-4\lambda}-(2+4\lambda^2)\,e^{-2\lambda}])^{-1}$ can be expanded in an infinite convergent series by means of the binomial theorem for every value of $\lambda\in(0,\infty)$. This is true since

$$|e^{-4\lambda}-(2+4\lambda^2)\,e^{-2\lambda}|<1 \quad \forall\lambda\in(0,\infty),$$

that is for every λ in the range of integration of (6.32).

If the expansion is performed for $\chi>\psi$, the resulting terms can be expressed as

$$f^z_{zx}(\lambda,\chi,\psi)=\frac{(2\nu-1)}{2}\lambda\,e^{-\lambda(\chi-\psi)}-\frac{\lambda^2}{2}(\chi-\psi)\,e^{-\lambda(\chi-\psi)}$$

$$+\text{terms of the form } \lambda^n\,e^{-\lambda p(\chi-\psi,\psi)}\ \forall n=1,2,3,\ldots,$$

where the $p(\chi-\psi,\psi)$ are affine functions of $(\chi-\psi)$ and ψ, such that $p>0$ $\forall\chi,\psi\in[0,1]$ and in particular $\lim_{\chi\to\psi}p(\chi-\psi,\psi)\neq 0$.

The stress, thus can be expressed as

$$\sigma^z_{zx}=\frac{1}{4\pi(1-\nu)h^2}\frac{x}{R}\left\{\frac{(2\nu-1)}{2}\int_{0^+}^{\infty}\lambda\,e^{-\lambda(\chi-\psi)}J_1\left(\lambda\frac{R}{h}\right)d\lambda\right.$$

$$-\tfrac{1}{2}(\chi-\psi)\int_{0\pm}^{\infty}\lambda^2\,e^{-\lambda(\chi-\psi)}J_1\left(\lambda\frac{R}{h}\right)$$

$$\left.+\text{terms of the form}\int_{0^+}^{\infty}\lambda^n\,e^{-\lambda p(\chi-\psi,\psi)}J_1\left(\lambda\frac{R}{h}\right)d\lambda\right\},$$

or after integration:

$$\sigma^z_{zx}=\frac{1}{4\pi(1-\nu)}\frac{x}{R}$$

$$\times\left\{\frac{(2\nu-1)}{2}\frac{R}{\left[R^2+(z-H)^2\right]^{3/2}}-\frac{3}{2}\frac{R(z-H)^2}{\left[R^2+(z-H)^2\right]^{5/2}}\right\}$$

$$+\text{terms of the form}\frac{1}{8\pi(1-\nu)h^2}\frac{x}{R}\frac{(n+1)!R/h}{p^{n+2}(\chi-\psi,\psi)}$$

$$\times F\left(\frac{n+2}{2},\frac{n+3}{2};2;-\frac{(R/h)^2}{p^2(\chi-\psi,\psi)}\right),$$

where $F(\alpha,\beta;\gamma;\delta)$ is the hypergeometric function.

By letting $r = [R^2 + (z-H)^2]^{1/2}$, $\varphi = \cos^{-1}\frac{R}{r} = \sin^{-1}\left(\frac{z-H}{r}\right)$ and $\theta = \cos^{-1}\frac{x}{R} = \sin^{-1}\frac{y}{R}$, the above can be written as

$$\sigma^z_{zx} = \frac{\cos\varphi\cos\theta}{8\pi(1-\nu)r^2}\{(2\nu-1) - 3\sin^2\varphi\}$$

$$+\text{terms of the form}\frac{\cos\varphi\cos\theta}{8\pi(1-\nu)h^2}\frac{(n+1)!r/h}{p^{n+2}(r\sin\varphi,\psi)}$$

$$\times F\left(\frac{n+2}{2},\frac{n+3}{2};3;\frac{r^2\cos^2\varphi}{h^2p^2(r\sin\varphi,\psi)}\right),$$

taking the limit as $r \to 0$, and recalling that $\lim_{r\to 0} p(r\sin\varphi,\psi) \neq 0$, we get

$$\sigma^z_{zx} = \frac{\cos\varphi\cos\theta}{8\pi(1-\nu)r^2}[(2\nu-1) - 3\sin^2\varphi], \quad r \to 0.$$

The above expression is identical to the one predicted by Kelvin's solution for a unit concentrated load in the z direction. Proof of the equivalent result for all other stresses and displacements follows in an entirely equivalent way.

b. Numerical evaluation of the proposed solution

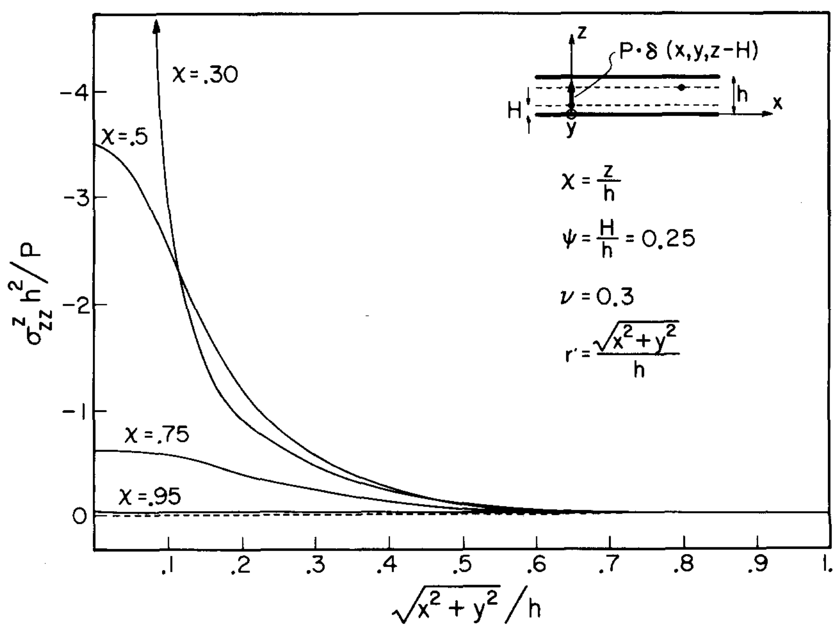

Fig. 2. Variation of the normalized stress $\sigma^z_{zz}h^2/P$ versus the normalized in-plane distance $r' = \sqrt{x^2+y^2}/h$.

Examples demonstrating some of the features of the three-dimensional solution are presented in Figs 2, 3 and 4. A point load along the z-direction was applied at a distance $0.25h$ from the lower surface of the layer. The variation of the σ_{zz}^{z} componment of the stresses with respect to the normalized in-plane distance $r' = \sqrt{x^2+y^2}/h$ measured from the point of application of the load is shown for the cases of $z = 0.95h$, $z = 0.75h$, $z = 0.5h$ and $z = 0.3h$. As expected, as $r \to 0$ ($z \to 0.25h$, $r' \to 0$) the stresses reproduce the singular behavior of the Kelvin state.

Figures 3 and 4 show the variation of the same stress component along the thickness of the plate for different values of the normalized in-plane distance $\sqrt{x^2+y^2}/h$ measured from the applied load. At distances close to the load (see Fig. 4, $\sqrt{x^2+y^2}/h = 0.05$), the stress changes rapidly from tensile to compressive as the plane of application of the load ($z = 0.25h$) is traversed. As the distance from the load is increased, the tensile portion of the thickness variation diminishes and eventually disappears. It is also worth noting that for distances greater that $0.5h$ the thickness variation becomes symmetrically shaped despite of the fact that the problem in non-symmetric in the thickness

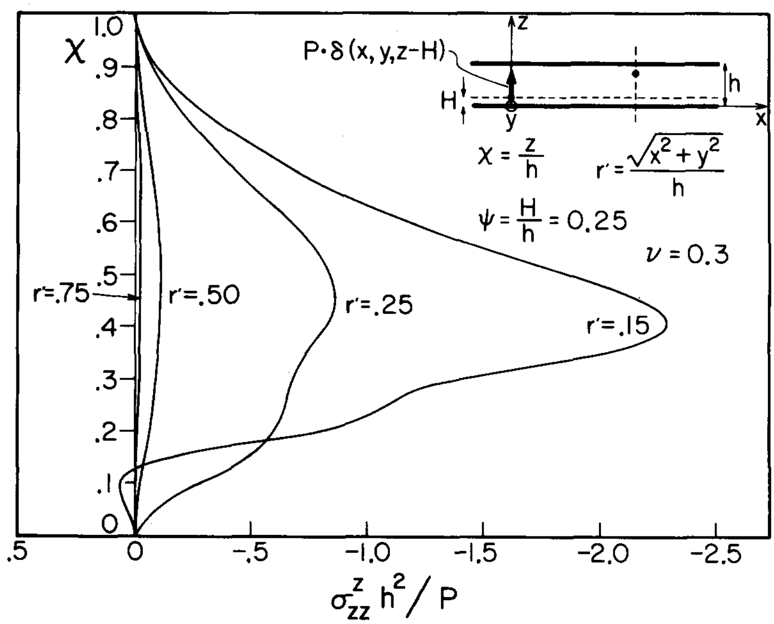

Fig. 3. Variation of the normalized stress $\sigma_{zz}^{z}h^2/P$ versus the normalized distance $\chi = z/h$ measured from the lower surface of the layer. Different curves correspond to $r' = 0.75$, $r' = 0.50$, $r' = 0.25$, $r' = 0.15$.

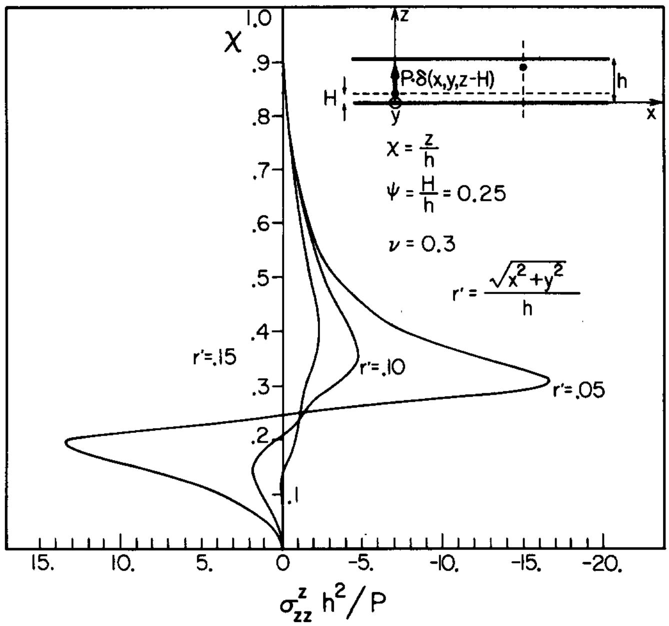

Fig. 4. Variation of the normalized stress $\sigma_{zz}^{z}h^2/P$ versus the normalized distance $\chi = z/h$ measured from the lower surface of the layer. Different curves correspond to $r' = 0.15$, $r' = 0.10$, $r' = 0.05$.

direction, suggesting that the decay length for the three-dimensional Saint-Venant problem is of the order of half the plate thickness.

7. Multilayered medium

If the system is a multilayered stack composed of N single layers, as shown in Fig. 5, we may write for the *k-th* layer:

$$\bar{a}_k^*(z_k) = \boldsymbol{T}_k(z_k)\bar{a}_k^*(0) + \overline{\boldsymbol{R}}_k(z_k), \tag{7.1}$$

where $\boldsymbol{T}_k(z_k)$ is given by (4.3) and E/E^* is replaced by E_k/E^*.

The *initial state vector* $\bar{a}_k^*(0)$ is thus related to the *end state vector* $\bar{\boldsymbol{a}}^*(h_k)$ through

$$\bar{a}_k^*(h_k) = \boldsymbol{T}_k(h_k)\bar{a}_k^*(0) + \overline{\boldsymbol{R}}_k(h_k), \tag{7.2}$$

where h_k is the thickness of the *k-th* layer.

Matrix $\boldsymbol{T}_k(z_k)$ has the properties defined in [9]; i.e., cross-symmetry, determinant 1, and for two layers of the same material

$$\boldsymbol{T}(h_1 + h_2) = \boldsymbol{T}(h_1)\boldsymbol{T}(h_2).$$

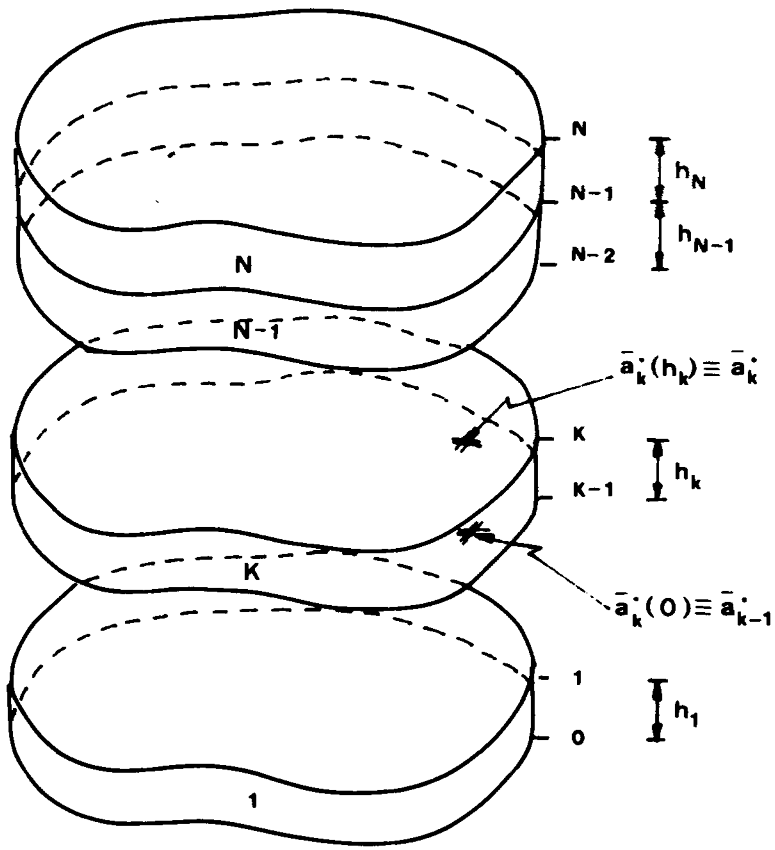

Fig. 5. System of *N* layers.

By knowing $\bar{a}_k^*(0)$, $\text{i}\bar{a}_k^*(z_k)$ is obtained from (7.1); the rest of the components can be inferred from (3.7) by

$$\bar{\boldsymbol{b}}_k(z_k) = \bar{\boldsymbol{B}}_k^*(z_k)\bar{a}_k^*(z_k) = \bar{\boldsymbol{B}}_k^*(z_k)\boldsymbol{T}_k(z_k)\bar{a}_k^*(0) + \boldsymbol{B}_k^*(z_k)\bar{\boldsymbol{R}}_k(z_k), \quad (7.3)$$

where $\bar{\boldsymbol{B}}^*$ is replaced by $\bar{\boldsymbol{B}}_k^*$, ν by ν_k and E/E^* by E_k/E^*.

By using (3.1), (3.5a) and (3.5b), the stresses and displacements in the kth layer will be given by

$$a_k^*(x, y, z_k) = \boldsymbol{F}^{-1}[\bar{a}_k^*(z_k)], \quad b_k(x, y, z_k) = \boldsymbol{F}'^{-1}[\bar{b}_k(z_k)]. \quad (7.4)$$

a. Method of transfer matrices

In the N-layers stack we can apply the continuity conditions

$\bar{a}_{k+1}^*(0) = \bar{a}_k^*(h_k) = \bar{a}_k$, $(\forall k = 1, 2, \ldots, N-1)$, and taking into account the expression (7.2), which now reads

$$\bar{a}_k^* = \boldsymbol{T}_k \bar{a}_{k-1}^* + \overline{\boldsymbol{R}}_k, \quad (\forall k = 1, 2, \ldots, N), \tag{7.5}$$

we may obtain

$$\bar{a}_N^* = \boldsymbol{T}_N \boldsymbol{T}_{N-1} \ldots \boldsymbol{T}_1 \bar{a}_0^* + \boldsymbol{T}_N \boldsymbol{T}_{N-1} \ldots \boldsymbol{T}_2 \overline{\boldsymbol{R}}_1 + \cdots + \boldsymbol{T}_N \overline{\boldsymbol{R}}_{N-1} + \overline{\boldsymbol{R}}_N, \tag{7.6}$$

an equation which relates the six components of the *end state vector* of the N-layer system $\bar{a}_N^*(h_k)$ with the six ones of the *initial state vector* $\bar{a}_0^*(0)$. In a well-posed problem, six of these twelve components will be known (all the stresses, in a problem with natural boundary conditions specified; all the displacements, in a problem with essential boundary conditions specified; some of the stresses and some of the displacements, in a problem with mixed boundary conditions). Once we have solved for the unknowns of system (7.6), we shall be able to able to obtain the intermediate *state vector* by using (7.5) and the rest of stress components by (7.3).

b. Method of flexibility matrices

The continuity condition will read:

$$\left.\begin{array}{l} \bar{u}_k^*(h_k) \equiv \bar{u}_{k+1}^*(0) \equiv \bar{u}_k^* \\ \bar{\sigma}_k(h_k) \equiv \bar{\sigma}_{k+1}(0) \equiv \bar{\sigma}_k \end{array}\right\} \quad (\forall k = 1, 2, \ldots, N-1);$$

and from (4.5)

$$\begin{pmatrix} \bar{u}_{k-1}^* \\ \bar{u}_k^* \end{pmatrix} = \begin{pmatrix} N_{11_k} & N_{12_k} \\ N_{21_k} & N_{22_k} \end{pmatrix} \begin{pmatrix} \bar{\sigma}_{k-1} \\ \bar{\sigma}_k \end{pmatrix} + \begin{pmatrix} M_{11_k} & 0 \\ M_{21_k} & 1 \end{pmatrix} \begin{pmatrix} \overline{R}_{1_k} \\ \overline{R}_{2_k} \end{pmatrix}.$$

From the above, the following matrix difference equation is obtained:

$$N_{21_k} \bar{\sigma}_{k-1} + \left(N_{22fk} - N_{11_{k+1}}\right) \bar{\sigma}_k - N_{12_{k+1}} \bar{\sigma}_{k+1} = M_{11_{k+1}} \overline{R}_{1_{k+1}} - M_{21_k} \overline{R}_{1_k} - \overline{R}_{2_k},$$
$$(\forall k = 1, 2, \ldots, N-1).$$

8. Non-adhesive multilayered medium

In the case where only normal tractions are transferred between layers, the following conditions hold:

$$\overset{\alpha}{\bar{\sigma}}_{zx_k}(h_k) = \overset{\alpha}{\bar{\sigma}}_{zx_k}(0) = \overset{\beta}{\bar{\sigma}}_{zy_k}(h_k) = \overset{\beta}{\bar{\sigma}}_{zy_k}(0) = 0 \quad (\forall k = 1, 2, \ldots, N).$$

From expression (7.2), we can infer a relationship among the displacements $\bar{u}_{y_k}^{\beta *}(0)$, $\bar{u}_{x_k}^{\alpha *}(0)$ with the stress $\bar{\sigma}_{zz_k}(0)$ and the displacement $\bar{u}_{z_k}^*(0)$. By considering only the equation relating $\bar{\sigma}_{zz_k}(h_k)$ and $\bar{u}_{z_k}^*(h_k)$, we can write

$$\begin{pmatrix} \bar{\sigma}_{zz}(h_k) \\ \bar{u}_z^*(h_k) \end{pmatrix}_k = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}_k \begin{pmatrix} \bar{\sigma}_{zz}(0) \\ \bar{u}_z^*(0) \end{pmatrix}_k - \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}_k \begin{pmatrix} \overline{R}_{xx} \\ \overline{R}_{zy} \end{pmatrix}_k + \begin{pmatrix} \overline{R}_{zz} \\ \overline{R}_z \end{pmatrix}_k,$$

or in compact form

$$\mathscr{J}_k^*(h_k) = \mathscr{T}_k(h_k)\mathscr{J}_k^*(0) + \bar{\mathscr{R}}_k(h_k), \tag{8.1}$$

where $\bar{R}_{ij}$ are the components of the vector $\bar{R}_k(h_k)$ defined as

$$\bar{R}_k(h_k) = \left(\bar{R}_{zz}, \bar{R}_{zx}, \bar{R}_{zy}, \bar{R}_y, \bar{R}_x, \bar{R}_z\right)_k^T,$$

where the matrices $\mathscr{T}_k(h_k)$ and column vectors $\bar{\mathscr{R}}_k(h_k)$ are given by

$$\mathscr{T}_k(h_k) = \begin{pmatrix} A_{11} & Q_{12} \\ A_{21} & A_{22} \end{pmatrix}_k; \quad \bar{\mathscr{R}}_k(h_k) = -\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}_k \begin{pmatrix} \bar{R}_{zx} \\ \bar{R}_{zy} \end{pmatrix}_k + \begin{pmatrix} \bar{R}_{zz} \\ \bar{R}_z \end{pmatrix}_k,$$

with the elements A_{ij} and B_{ij} being

$$A_{11} = \frac{\cosh\left(\lambda \frac{z}{h}\right)}{\tanh\left(\lambda \frac{z}{h}\right) + \lambda \frac{z}{h}} \left\{\tanh\left(\lambda \frac{z}{h}\right) + \left[1 - \tanh^2\left(\lambda \frac{z}{h}\right)\right]\left(\lambda \frac{z}{h}\right)\right\},$$

$$A_{12} = \frac{\cosh\left(\lambda \frac{z}{h}\right)}{\tanh\left(\lambda \frac{z}{h}\right) + \lambda \frac{z}{h}} \frac{\lambda}{2(1-\nu^2)} \frac{E}{E^*}$$

$$\times \left\{\left(\lambda \frac{z}{h}\right)^2 \left[1 - \tanh^2\left(\lambda \frac{z}{h}\right)\right] - \tanh^2\left(\lambda \frac{z}{h}\right)\right\},$$

$$A_{21} = \frac{\cosh\left(\lambda \frac{z}{h}\right)}{\tanh\left(\lambda \frac{z}{h}\right) + \lambda \frac{z}{h}} \frac{-2(1-\nu^2)}{\lambda} \frac{E^*}{E} \tanh^2\left(\lambda \frac{z}{h}\right),$$

$$A_{22} = \frac{\cosh\left(\lambda \frac{z}{h}\right)}{\tanh\left(\lambda \frac{z}{h}\right) + \lambda \frac{z}{h}} \left\{\tanh\left(\lambda \frac{z}{h}\right) + \left[1 - \tanh^2\left(\lambda \frac{z}{h}\right)\right]\left(\lambda \frac{z}{h}\right)\right\},$$

$$B_{11} = +\frac{1}{\tanh\left(\lambda \frac{z}{h}\right) + \lambda \frac{z}{h}} \alpha^* \frac{z}{h} \tanh\left(\lambda \frac{z}{h}\right),$$

$$B_{12} = -\frac{1}{\tanh\left(\lambda \frac{z}{h}\right) + \lambda \frac{z}{h}} \beta^* \frac{z}{h} \tanh\left(\lambda \frac{z}{h}\right),$$

$$B_{21} = \frac{1}{\tanh\left(\lambda \frac{z}{h}\right) + \lambda \frac{z}{h}} \frac{\alpha^*(1+\nu)}{\lambda^2} \frac{E^*}{E} \left\{\lambda \frac{z}{h} + (2\nu - 1)\tanh\left(\lambda \frac{z}{h}\right)\right\},$$

$$B_{22} = \frac{1}{\tanh\left(\lambda \frac{z}{h}\right) + \lambda \frac{z}{h}} \frac{\beta^*(1+\nu)}{\lambda^2} \frac{E^*}{E} \left\{\lambda \frac{z}{h} + (2\nu - 1)\tanh\left(\lambda \frac{z}{h}\right)\right\}.$$

Now, the matrix $\mathscr{T}_k$ is cross-symmetric but generally $\mathscr{T}(h_1+h_2) \neq \mathscr{T}(h_1)\mathscr{T}(h_2)$, due to the jumps of the displacements $\bar{u}_x^{\alpha *}$ and $\bar{u}_y^{\beta *}$ at the interlayer surfaces.

Similarly to equation (7.6), the continuity conditions will yield

$$\not{p}_N^* = \mathscr{T}_N \mathscr{T}_{N-1} \ldots \mathscr{T}_1 \not{p}_0^* + \mathscr{T}_N \mathscr{T}_{N-1} \ldots \mathscr{T}_2 \bar{\mathscr{R}}_{N-1} \ldots \mathscr{T}_2 \bar{\mathscr{R}}_1 + \cdots + \mathscr{T}_N \bar{\mathscr{R}}_{N-1} + \bar{\mathscr{R}}_N. \tag{8.2}$$

Once the unknowns of system (8.2) have been solved, the complete *state vector* can be obtained from

$$\begin{pmatrix} \bar{u}_y^{\beta *}(h_k) \\ \bar{u}_x^{\alpha *}(h_k) \end{pmatrix}_k = \begin{pmatrix} t_{41} & t_{42} & t_{43} & t_{44} & t_{45} & t_{46} \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} & t_{56} \end{pmatrix}_k \bar{a}_k^*(0) + \begin{pmatrix} \bar{R}_y \\ \bar{R}_x \end{pmatrix}_k,$$

where t_{ij} are the elements of the matrix $T_k(h_k)$, given in (4.3) for which $z = h_k$, $\nu = \nu_k$ and $E = E_k$. Knowing $\bar{a}_k^*(0)$, the *state vector* $\bar{a}^*(z_k)$ can be calculated from (7.1), and the rest of the components will be completed using (7.3). The inverse transforms of the *state vectors* are given by (7.4).

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Appendix A

The stress and displacement functions in the physical space obtained from (5.3) contain integrals of the following structure:

$$I = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\lambda)\, e^{-i(\alpha x + \beta y)}\, d\alpha\, d\beta, \quad \lambda = h\sqrt{\alpha^2 + \beta^2}$$

Depending on $F(\lambda)$, these integrals can be expressed in cylindrical coordinates as follows:

a.1. Case $F(\lambda) = f(\lambda)$,

$$I = \frac{2\pi}{h^2} \int_{\lambda=0}^{\lambda=\infty} \lambda f(\lambda) J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h} \right) d\lambda;$$

a.2 Case $F(\lambda) = \alpha f(\lambda)$,

$$I = \frac{1}{h^3} \frac{-2\pi\, ix}{\sqrt{x^2+y^2}} \int_{\lambda=0}^{\lambda=\infty} \lambda^2 f(\lambda) J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h} \right) d\lambda;$$

a.3 Case $F(\lambda)=\beta f(\lambda)$,

$$I=\frac{1}{h^3}\frac{-2\pi\,\mathrm{i}y}{\sqrt{x^2+y^2}}\int_{\lambda=0}^{\lambda=\infty}\lambda^2 f(\lambda)J_1\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda;$$

a.4 Case $F(\lambda)=\alpha^2 f(\lambda)$,

$$I=\frac{\pi}{h^4}\int_{\lambda=0}^{\lambda=\infty}\lambda^3 f(\lambda)J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda$$

$$+\frac{\pi}{h^4}\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda^3 f(\lambda)J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda;$$

a.5 Case $F(\lambda)=\beta^2 f(\lambda)$,

$$I=\frac{\pi}{h^4}\int_{\lambda=0}^{\lambda=\infty}\lambda^3 f(\lambda)J_0\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda$$

$$-\frac{\pi}{h^4}\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda^3 f(\lambda)J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda;$$

a.6 Case $F(\lambda)=\alpha\beta f(\lambda)$,

$$I=-\frac{2\pi}{h^4}\frac{xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda^3 f(\lambda)J_2\left(\frac{\lambda\sqrt{x^2+y^2}}{h}\right)\mathrm{d}\lambda.$$

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