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REEXAMINATION OF JUMPS ACROSS QUASISTATICALLY PROPAGATING SURFACES UNDER GENERALIZED PLANE STRESS IN ANISOTROPICALLY HARDENING ELASTIC-PLASTIC SOLIDS

by

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1. INTRODUCTION

A variety of problems of physical interest involving the deformation of elastic-plastic solids may require the admission of discontinuities in the gradients of stresses and velocities (weak discontinuities) or in these quantities themselves (strong discontinuities). Such discontinuities may occur within regions which are currently deforming plastically or at elastic-plastic boundaries. These possibilities have received wide attention for rigid - perfectly plastic solids in plane strain (Hill [3]) and in generalized plane stress (Hill [1]) in the presence of either the isotropic Huber-von Mises or Tresca yield conditions in the plastic range. It is well-known that for such solids, strong discontinuities in stress and velocity cannot be simultaneously present, and that velocity jumps occur across characteristic surfaces. It has been noted by Hill [1] that when a rigid-plastic generalized plane-stress theory is employed in the study of the extension of thin plates, two types of strong discontinuities must be considered. These arise because of the neglect of elastic deformation and the averaged nature of generalized plane stress. A consideration of the second of these factors has led to the mathematical idealization of the experimentally observed phenomenon of localized necking in thin sheets (Nadai [8]).

In a recent paper [6], Drugan and Rice investigated strong discontinuities across quasistatically propagating surfaces in elastic-plastic solids under general three-dimensional conditions when all displacement components are assumed to be continuous. One important conclusion of the work reported in [6] is that all stress components are always continuous, a result that follows from certain material stability postulates.

Pan [7] has also discussed quasistatically moving strong discontinuities for elastic-perfectly plastic Huber-von Mises materials under generalized plane stress. He assumes that a strong discontinuity can be replaced by a transition layer of elastic material in which all stress components are assumed to vary continuously. He subsequently demonstrates full continuity of stresses across propagating surfaces, by using the specific nature of the Huber-von Mises locus and arriving at a contradiction.

In the present work quasistatic discontinuities are reexamined for the more general case of an anisotropic hardening solid using an integral form of the maximum plastic work inequality and the usual assumptions in the theory of generalized plane stress (Section 2). It is demonstrated in Section 5 that the use of the maximum plastic work inequality leads to full stress continuity for a broad class of solids which includes some hardening materials and anisotropic behavior. Pan's assumptions and the limitations of his approach are discussed in Section 5. A complete analysis of all possible velocity jumps including sliding discontinuity and localized necks is carried out in Section 6 with some generality in constitutive behavior. These results are then specialized for the elastic-perfectly plastic Huber-von Mises solid. A simplified expression is obtained for positive plastic work accumulation due to the passage of a discontinuity surface.

One important application of the results of this work is to quasistatic crack growth problems in elastic-plastic solids under conditions of generalized plane stress (Rice [9]). The stress and velocity fields near the tip of a growing crack may be constructed by assembling plastic and elastic unloading sectors and by satisfying proper matching conditions, boundary conditions and material stability postulates. Some restrictions arising from the jump conditions on this assembly are pointed out in Section 7.

2. THE GENERALIZED PLANE-STRESS PROBLEM

Consider an elastic-plastic body occupying an open cylindrical region R of height 2h (see Figure 1).

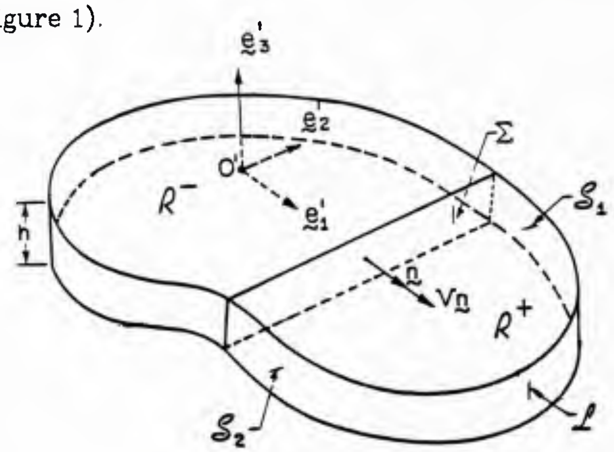

Figure 1. Elastic-plastic body with discontinuity surface Σ.

Let the boundary ∂R of the above region be composed of two traction free planar surfaces S_1 and S_2 and a lateral surface L.

Consider further a fixed orthonormal coordinate system $\{o', \underset{\sim}{e}'_1, \underset{\sim}{e}'_2, \underset{\sim}{e}'_3\}$ such that $\underset{\sim}{e}'_3$ is parallel to the generators of R.

Generalized plane stress conditions require that the height of the cylinder (also referred later as the thickness of the cylindrical plate) be small as compared with any other dimension of the cylinder, and that the prescribed tractions $\underset{\sim}{t}$ be such that:

$$\underset{\sim}{t} = \underset{\sim}{0} \quad \text{or} \quad \sigma_{3i} = 0 \quad \text{on } S_1 \text{ and } S_2$$

and

$$t_3 = 0, \quad t_\alpha = t^*_\alpha(x_1, x_2) \text{ on L.} \tag{2.1}$$

Here σ_{ij} are the components of the symmetric Cauchy stress tensor. Greek subscript have the range 1, 2 while Latin subscipts take the values 1, 2, and 3. (This convention will be adopted through the following development.)

In what follows field quantities such as $\underset{\sim}{\sigma}$, $\underset{\sim}{\varepsilon}$, $\underset{\sim}{u}$, and $\underset{\sim}{v}$ will represent thickness averages of the stress and strain tensors and the displacement and velocity vectors respectively. It is also assumed that,

$$\sigma_{3i} \equiv 0 \quad \text{on } R. \tag{2.2}$$

The above assumptions result in solutions of the generalized plane stress problem which in general will not satisfy the exact three dimensional field equations as discussed in detail by Timoshenko [5], and Hill [3]. This is because some of the compatibility equations are not generally satisfied and errors are involved in using the averaged quantities in the constitutive law and the yield condition. However if the plate thickness is sufficiently small, the generalized plane stress solution is expected to provide an accurate approximation.

Let Σ be a planar surface, parallel to the $x_2'-x_3'$ plane dividing the region R in two open subregions R^+ and R^- such that

$$R = R^+ \cup R^- \cup \Sigma$$

We will define the normal $\underset{\sim}{n}(\overset{\circ}{\underset{\sim}{x}})$ to Σ at a point $\overset{\circ}{\underset{\sim}{x}} \in \Sigma$ as the outward normal of the closed subregion $\bar{R}^-(\bar{R}^\pm \equiv R^\pm \cup \Sigma)$ at the same point $\overset{\circ}{\underset{\sim}{x}}$.

In what follows the surface Σ will be viewed as a potential surface of strong discontinuities (discontinuities in stresses and strains) and will be allowed to translate quasistatically with a normal velocity $V\underset{\sim}{n}$.

Since the approximate theory of generalized plane stress treats the thickness of the plate as vanishingly small, Hill [1] points out that every quantity whose gradient is of order (1/h) in a zone of breadth comparable to h *should be modeled as a discontinuity*. Thus the experimentally observed formation of necks (Nadai [8]) in thin plates subjected to tension (rapid variation of thickness of the plate in narrow zones) would be modeled as discontinuities in the out-of-plane displacement component u_3.

The jump in a field quantity $\underset{\sim}{g}(\underset{\sim}{x})$, across the surface Σ, will be denoted by:

$$\left.\begin{aligned} &[\underset{\sim}{g}] \equiv \underset{\sim}{g}^{+}(\overset{o}{\underset{\sim}{x}}) - \underset{\sim}{g}^{-}(\overset{o}{\underset{\sim}{x}}) \quad \text{where} \\ &\underset{\sim}{g}^{\pm}(\overset{o}{\underset{\sim}{x}}) = \lim_{\varepsilon\to 0} \underset{\sim}{g}(\overset{o}{\underset{\sim}{x}} \pm \varepsilon \underset{\sim}{n}(\overset{o}{\underset{\sim}{x}})) \quad \overset{o}{\underset{\sim}{x}} \in \Sigma \quad \text{and} \quad \varepsilon > 0 \end{aligned}\right\} \tag{2.3}$$

3. SMOOTHNESS CONSIDERATIONS

All field quantities will be referred to, with respect to an orthonormal frame $\{O, \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ translating with the surface Σ and such that $O \in \Sigma$, $\underset{\sim}{e}_3 = \underset{\sim}{e}_3'$ and $\underset{\sim}{e}_1 = \underset{\sim}{n}$; see Figure 2.

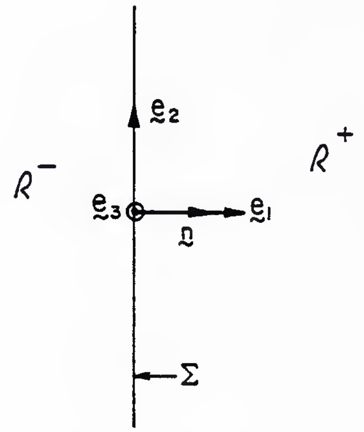

Figure 2. Local coordinate system translating with the surface.

In-plane displacement components u_α are required to have the following smoothness properties:

$$\begin{aligned} &u_\alpha \in C(R) \quad \text{and} \\ &u_\alpha \in C^1(R-\Sigma) \end{aligned} \tag{3.1}$$

with the understanding that $\dfrac{\partial u_\alpha}{\partial x_\beta}$ need not be continuous across Σ. Then according to the Hadamard compatibility relations [2] for jumps in the derivatives of a continuous function,

$$\left[\frac{\partial u_\alpha}{\partial x_\beta} \right] = \lambda_\alpha n_\beta \quad \text{on} \quad \Sigma \tag{3.2}$$

where λ_α are arbitrary functions of position on Σ. The out-of-plane displacement component u_3 will in general be allowed to suffer a jump across Σ, as discussed in Section 2. Thus:

$$u_3 \in C^1(R^+) \cup C^1(R^-) \tag{3.3}$$

with the understanding that on Σ, u_3 and its gradient need not be defined. On the other hand, $[u_3]$, the jump in the limiting value of u_3 from R^- to R^+ will be assumed to be a continuous and continuously differentiable function of position on Σ.

It is now possible to extend the Hadamard compatibility relations (3.2) for the treatment of jumps in the derivatives of discontinuous functions. This extension was first discussed by Thomas [4]. The following simpler version was later provided by Hill [2],

$$\left[\frac{\partial u_3}{\partial x_i} \right] = \lambda_3 n_i + \frac{\partial \varphi}{\partial x_i} \quad \text{on} \ \Sigma \tag{3.4}$$

where λ_3 is an arbitrary function of position on Σ and φ is an arbitrary continuous function together with its gradient on Σ and in one neighborhood, say R^+, with the additional restriction that

$$\varphi = [u_3] \ \text{on} \ \Sigma.$$

One choice of φ in $\overline{R}^+$ would be to consider φ continued analytically along the normals. Any other choice would merely change λ_3 which is given by

$$\lambda_3 = [\nabla \ u_3 \cdot \underset{\sim}{n}] - \nabla \ \varphi \cdot n$$

Relations (3.2) and (3.4) allow definition of jumps in the strains across Σ consistent with the assumptions of the approximate theory of generalized plane stress.

Within the contexts of a small strain formulation,

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3.5}$$

and the jumps in the in-plane strain component $\varepsilon_{\alpha\beta}$ can be expressed by (3.2) as:

$$[\varepsilon_{\alpha\beta}] = \frac{1}{2}(\lambda_\alpha n_\beta + \lambda_\beta n_\alpha) \quad \text{on } \Sigma \tag{3.6}$$

On the other hand, the jump in the out of plane strain component ε_{33} can be expressed by (3.2) and (3.4) as:

$$[\varepsilon_{33}] = \lambda_3 n_3 + \frac{\partial\varphi}{\partial x_3} \quad \text{on } \Sigma \tag{3.7}$$

where $\varphi \in C^1(\overline{R}^+)$ and $\varphi=[u_3]$ on Σ.

4. MATERIAL IDEALIZATION

Within the context of the small-strain flow theory of plasticity, the total strain rate tensor can be decomposed into elastic and plastic parts:

$$\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{\varepsilon}}^{e} + \dot{\underset{\sim}{\varepsilon}}^{P} \quad \text{on } R \tag{4.1}$$

where the dot denotes differentiation with respect to time. The elastic strain rate tensor $\dot{\underset{\sim}{\varepsilon}}^{e}$ is related to the stress rate tensor $\dot{\underset{\sim}{\sigma}}$ through a constant, positive definite four-tensor $\underset{\sim}{S}$ (the inverse of the elasticity tensor $\underset{\sim}{C}$). $\underset{\sim}{S}$ is assumed to possess the usual major and minor symmetries. For an anisotropic elastic-plastic solid, $\dot{\underset{\sim}{\varepsilon}}^{e}$ is given by:

$$\dot{\underset{\sim}{\varepsilon}}^{e} = \underset{\sim}{S}\,\dot{\underset{\sim}{\sigma}} \quad \text{on } R \tag{4.2}$$

The condition for the existence of non-zero plastic strain rates $\dot{\underset{\sim}{\varepsilon}}^{P}$, called the yield condition, is formulated on the basis of a scalar valued function of the stresses and the total accumulated equivalent plastic strain $(\bar{\varepsilon}^P)$ denoted by $f(\underset{\sim}{\sigma}, \bar{\varepsilon}^P)$ and on an appropriate definition of loading, unloading or neutral loading. The condition $f(\underset{\sim}{\sigma}, \bar{\varepsilon}^P)=0$ defines a hypersurface in 6-dimensional stress space, called the yield surface. In this work, attention will be focussed on the particular class of materials obeying Drucker's stability postulate. A particular form of Drucker's postulate, known as the maximum plastic work inequality can be expressed as:

$$(\underset{\sim}{\sigma} - \underset{\sim}{\sigma}^*) \cdot \dot{\underset{\sim}{\varepsilon}}^P \geq 0 \tag{4.3}$$

$\forall f(\underset{\sim}{\sigma}, \bar{\varepsilon}^P)=0$, and $f(\underset{\sim}{\sigma}^*, \bar{\varepsilon}^P) \leq 0$. The two most important implications of the above postulate are the following:

a. The hypersurface $f(\underset{\sim}{\sigma}, \bar{\varepsilon}^P)=0$ is convex.

b. The plastic strain rate $\dot{\underset{\sim}{\varepsilon}}^P$ is normal to the yield surface, and the flow rule takes the following form:

$$\dot{\underset{\sim}{\varepsilon}}^P = \dot{\lambda} \underset{\sim}{P} \tag{4.4}$$

where $\dot{\lambda} \geq 0$ and $\underset{\sim}{P} = \nabla_{\underset{\sim}{\sigma}} f$. $\dot{\lambda}$ and $\underset{\sim}{P}$ are scalar valued and symmetric tensor valued functions of $\underset{\sim}{\sigma}$ respectively. In the following section, an integral form of (4.3) will be used in conjunction with equations (4.1) and (4.2) as well as the compatibility conditions for the jumps in total strains (3.6), (3.7) to define the jumps in the stresses and the plastic strains produced during the passage of a discontinuity Σ through a material point.

5. STRESS CONTINUITY ACROSS THE PROPAGATING SURFACE

In this section it will be demonstrated that all stress components are continuous across the surface Σ propagating quasistatically through the thin plate. It will be shown that this is true even if the out of plane displacement u_3 suffers a discontinuity across Σ. The following proof is based on the maximum plastic work inequality and the positive definiteness of $\underset{\sim}{S}$. It is an extension of the proof given by Drugan and Rice [6] for the general three-dimensional case. In the present analysis, only the in-plane displacement components u_α are assumed continuous, and the proof is adapted to suit the assumptions of the theory of generalized plane stress. Also, unlike the discussion by Pan [7] and consistent with the assumptions of generalized plane stress, [1] our discussion treats necks as jumps and *not* as narrow transition layers.

If inertia terms are neglected, the balance of linear momentum requires that across the quasistatically moving surface Σ the traction should be continuous. Thus

$$[\,t_\alpha] = [\sigma_{\alpha\beta}\, n_\beta\,] = 0 \quad \text{on}\ \Sigma.$$

With respect to the local orthonormal coordinate frame $\{O,\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}$ moving with Σ, $n_i=\delta_{1i}$ and the above conditions become:

$$[\sigma_{1\alpha}] = 0 \quad \text{on} \quad \Sigma \tag{5.1}$$

Equations (2.2) and (5.1) imply that the only stress component that can suffer a non-trivial jump is σ_{22}. The plastic work W^P accumulated discontinuously at a material point due to the passage of the surface Σ is given by:

$$W^P = \int_{\underset{\sim}{\varepsilon}^{P+}}^{\underset{\sim}{\varepsilon}^{P-}} \underset{\sim}{\sigma} \cdot d\underset{\sim}{\varepsilon}^{P}. \tag{5.2}$$

It should be observed here that some error is involved in using the averaged stress and strain quantities of generalized plane stress in the above integral. The above integral is evaluated according to the assumptions of Section 2. On applying equation (2.2), the plastic work accumulation in (5.2) reduces to,

$$W^P = \int_{\varepsilon_{\alpha\beta}^{P+}}^{\varepsilon_{\alpha\beta}^{P-}} \sigma_{\alpha\beta}\, d\,\varepsilon_{\alpha\beta}^{P}.$$

(5.3)

Using (5.1) the above becomes:

$$W^P = -\sigma_{11}[\varepsilon_{11}^P] - 2\sigma_{12}[\varepsilon_{12}^P] + \int_{\varepsilon_{22}^{P+}}^{\varepsilon_{22}^{P-}} \sigma_{22}\, d\varepsilon_{22}^P \tag{5.4}$$

Also, by using the fact that $n_\beta = \delta_{1\beta}$, (3.6) implies that:

$$[\varepsilon_{22}] = 0 \text{ on } \Sigma \text{ or } [\varepsilon_{22}^e] = -[\varepsilon_{22}^P] \text{ on } \Sigma \tag{5.5}$$

By setting $d\underset{\sim}{\varepsilon}^P = d\underset{\sim}{\varepsilon} - d\underset{\sim}{\varepsilon}^e$, using the continuity of ε_{22} across Σ (5.5) and integrating by parts, (5.4) becomes

$$W^P = -\sigma_{11}[\varepsilon_{11}^P] - 2\sigma_{12}[\varepsilon_{12}^P] - \int_{\varepsilon_{22}^{e+}}^{\varepsilon_{22}^{e-}} \sigma_{22}\, d\varepsilon_{22}^e. \tag{5.6}$$

The integral in (5.6) can now be evaluated by using equations (2.2), (5.1), and the constitutive law, to give:

$$\int_{\varepsilon_{22}^{e+}}^{\varepsilon_{22}^{e-}} \sigma_{22}\, d\varepsilon_{22}^e = -\frac{1}{2} S_{2222}(\sigma_{22}^+ + \sigma_{22}^-)[\sigma_{22}] \tag{5.7}$$

In addition, from (2.2), (5.1), and (5.5),

$$[\varepsilon_{22}^P] = -[\varepsilon_{22}^e] = -S_{2222}[\sigma_{22}] \text{ on } \Sigma \tag{5.8}$$

Thus equations (5.6) and (5.7) give:

$$W^P = -\sigma_{11}[\varepsilon_{11}^P] - 2\sigma_{12}[\varepsilon_{12}^P] - \frac{1}{2}(\sigma_{22}^+ + \sigma_{22}^-)[\varepsilon_{22}^P]$$

or,

$$W^P = -\frac{1}{2}(\sigma_{ij}^+ + \sigma_{ij}^-)[\varepsilon_{ij}^P]. \tag{5.9}$$

The above equation is of the same form as the corresponding result obtained by Drugan and Rice [6] for the three-dimensional case. It should, however, be borne in mind that unlike the equivalent expression in [6], (5.9) was obtained by requiring that only the in plane displacement components are continuous and by using the assumptions of generalized plane stress. It should also be observed that the restrictions imposed on the path in stress space in the evaluation of the integral in (5.6) are the plane stress conditions and the continuity of tractions across Σ. This effectively implies a straight line path in stress space from σ_{22}^{+} to σ_{22}^{-}.

The integral form of the plastic work inequality (4.3) can, now be used by setting $\underset{\sim}{\sigma}^{*}=\underset{\sim}{\sigma}^{+}$ where $f(\underset{\sim}{\sigma}^{+},\bar{\varepsilon}^{P}) \leq 0$. Thus, $\underset{\sim}{\sigma}^{+}$ is constrained to remain always at or inside the yield surface during passage of Σ. Thus by using (2.2), (5.1), and (5.9):

$$\int_{\varepsilon_{22}^{P+}}^{\varepsilon_{22}^{P-}} (\sigma_{ij} - \sigma_{ij}^{+}) d\varepsilon_{ij}^{P} = -\frac{1}{2}(\sigma_{22}^{+} + \sigma_{22}^{-})[\varepsilon_{22}^{P}] + \sigma_{22}^{+}[\varepsilon_{22}^{P}] \geq 0$$

which by (5.8) gives

$$\frac{1}{2}[\sigma_{22}]\,[\varepsilon_{22}^{e}] \leq 0 \quad \text{or} \quad \frac{1}{2}([\sigma_{22}])^2\, S_{2222} \leq 0 \tag{5.10}$$

(5.10) now requires that $[\sigma_{22}] = 0$ since $S_{2222} > 0$.

Remarks

The following remarks are relevant:

1. Under generalized plane stress conditions, all stress components are continuous across the slowly propagating surface Σ even if the out of plane displacement u_3 suffers a discontinuity. Drugan and Rice [6] establishes *full* stress continuity under general three-dimensional conditions, with all displacement components assumed continuous.

2. The present discussion applies to general anisotropic elastic-plastic hardening solids obeying a flow rule of the associated type. The proof of full stress

continuity is based on an integral form of the maximum plastic work inequality and the positive definiteness of the elastic potential. The initial yield surface can be anisotropic but should be convex. All types of hardening in which the subsequent yield locus fully encloses all previous yield loci can be included.

3. An earlier discussion by Pan [7] is limited to elastic- ideally plastic solids of a Huber-von Mises type under generalized plane stress conditions. His argument, which does not make use of the maximum plastic work inequality, follows from Hill's [3] statement that the stress state from σ_{22}^{+} to σ_{22}^{-} can be bridged only by a succession of elastic states. This assumes a smooth variation of stresses in a "transition layer." Such an assumption is questionable for generalized plane stress since, as pointed out in Section 2, any field quantity whose gradient is $O(\frac{1}{h})$ in a zone of breadth comparable to h should be modelled as a discontinuity. Even if this assumption is accepted, Pan's argument clearly does not apply to arbitrary yield surfaces or general hardening solids. For instance, in elastic-perfectly plastic solids characterized by a Tresca yield condition when the neck (discontinuity in u_3) coincides with a principal stress direction and $\sigma_{11} = \pm\ \sigma_o$, the stress component σ_{22} can have *any* value between 0 and $\pm\ \sigma_o$ and still lie on the yield surface (Hill [3]). Hence $\dot{\lambda}$ in equation (4.4) is not necessarily zero in the transition from $\underset{\sim}{\sigma}^{+}$ to $\underset{\sim}{\sigma}^{-}$ (Hill [1]) and the argument fails. Also for any type of hardening solid, the consistency condition requires the stress state to lie on the yield surface during the process from $\underset{\sim}{\sigma}^{+}$ to $\underset{\sim}{\sigma}^{-}$ and no elastic unloading is possible.

6. DISCONTINUITIES IN STRAINS AND VELOCITIES

In this section, the earlier result pertaining to continuity of stresses across Σ will be used to provide restrictions on the nature of admissible jumps in strains and material particle velocities across Σ for a general anisotropic hardening solid. Attention will then be turned to plastically incompressible, generally anisotropic, elastic-perfectly plastic solids with smooth but otherwise arbitrary yield surfaces. Specialized results will be given for Huber-von Mises solids at the end of the discussion.

General Considerations:

The jumps in the in-plane velocity component v_α are given ([2], [6]) by:

$$[v_\alpha] = -V\left[\frac{\partial u_\alpha}{\partial x_1}\right] \text{ on } \Sigma \tag{6.1}$$

where $V\underset{\sim}{n}$ is the normal velocity of Σ. Making use of (3.2) and (3.6) the velocity jumps may be expressed as:

$$\begin{aligned} [v_1] &= -V[\varepsilon_{11}] \\ [v_2] &= -2V[\varepsilon_{12}] \end{aligned} \quad \text{on } \Sigma \tag{6.2}$$

Full stress continuity and (4.2) require the elastic part of the strains to be continuous across Σ

$$[\varepsilon^e_{ij}] = 0 \quad \text{on} \quad \Sigma \tag{6.3}$$

The above, and equation (5.8), therefore imply

$$[\varepsilon^p_{22}] = 0 \quad \text{on } \Sigma \tag{6.4}$$

As a result, the expression for the positive plastic work accumulation in (5.9) becomes

$$W^p = -\sigma_{11}[\varepsilon^p_{11}] - 2\sigma_{12}[\varepsilon^p_{12}] \geq 0 \tag{6.5}$$

and the jumps in the velocity components v_1 and v_2 are given by:

$$\begin{aligned}[v_1] &= -V\,[\varepsilon^P_{11}] \\ [v_2] &= -2V\,[\varepsilon^P_{12}]\end{aligned} \quad \text{on } \Sigma \tag{6.6}$$

The plastic work W^P can now be expressed in terms of velocity jumps as follows:

$$W^P = \frac{1}{V}(\sigma_{11}[v_1] + \sigma_{12}[v_2]) \geq 0 \tag{6.7}$$

No specific restrictions on the constitutive model other than the general assumptions made in Section 4 have been imposed in the derivation of equations (6.1)-(6.7).

For the specific class of plastically incompressible solids:

$$[\varepsilon^P_{33}] = -[\varepsilon^P_{11}] - [\varepsilon^P_{22}] \quad \text{on } \Sigma \tag{6.8}$$

which by use of equation (6.4) simplifies to:

$$[\varepsilon^P_{33}] = -[\varepsilon^P_{11}] \quad \text{on } \Sigma \tag{6.9}$$

Equation (6.9) serves to determine the jump in the out of plane plastic strain component ε^P_{33} in terms of the jump in the in-plane plastic strain component ε^P_{11} for plastically incompressible solids.

If the displacement component u_3 happens to be continuous across Σ as in [6], then ε_{33} and hence ε^P_{33} would also be continuous. Equations (6.8) and (6.6) will then imply that ε^P_{11} and v_1 should also be continuous across Σ. Thus it follows that for a plastically incompressible solid, when the surface Σ does not coincide with a neck (jump in u_3) only a sliding velocity discontinuity (jump in v_2) is permissible.

Elastic-Perfectly Plastic Solid

For such solids, the yield surface is represented by

$$f(\underset{\sim}{\sigma}) = 0 \quad \text{on } R \tag{6.10}$$

where $f(\underset{\sim}{\sigma})$ depends symmetrically on $\underset{\sim}{\sigma}$ and $\underset{\sim}{\sigma}^T$. It will also be assumed here that the yield surface is smooth (has a continuous normal)

Under such circumstances the flow rule takes the following form:

$$\dot{\underset{\sim}{\varepsilon}}^P = \dot{\lambda}\, \underset{\sim}{P} \quad \text{on } R \tag{6.11}$$

where $\dot{\lambda} \geq 0$ is an undetermined scalar function of position, and

$$\underset{\sim}{P}(\underset{\sim}{\sigma}) \equiv \nabla_{\underset{\sim}{\sigma}} f(\underset{\sim}{\sigma}) \quad \text{on } R \tag{6.12}$$

is a symmetric tensor-valued function of $\underset{\sim}{\sigma}$. Under conditions of generalized plane stress, equations (6.10) and (6.12) should be used in conjunction with the constraint (2.2). Inside regions which are currently deforming plastically, it can be shown from the two in-plane equilibrium equations, the yield condition and the plane stress assumption (2.2) that along stress characteristic directions the direct components of $P_{\alpha\beta}$ should vanish (Hill [3]).

It is also clear that, P_{ij} should be continuous across Σ from assumed smoothness of the yield surface and the requirement of full stress continuity. Then, from (6.11) the jumps in the plastic strain component ε_{ij}^P becomes:

$$[\varepsilon_{ij}^P] = -\eta\, P_{ij} \text{ on } \Sigma \tag{6.13}$$

where $\eta = \int_{\lambda^+}^{\lambda^-} d\lambda \geq 0$ is an undetermined scalar function of position on Σ.

Since $[\varepsilon_{22}^P] = 0$ across Σ, equation (6.13) implies that either $\eta=0$ or $P_{22}=0$ or both. If $\eta=0$, (6.13) requires all strain components to be continuous. Thus the ***necessary*** condition for non-trivial jumps in strains to exist across Σ is that P_{22} should vanish on Σ. In other words, Σ ***should coincide with a stress characteristic direction of its plastic side.***

This condition is less restrictive than the necessary condition for non-trivial jumps in the plastic strain components derived by Drugan and Rice [6] when *all* displacements were continuous across Σ. The corresponding necessary condition derived in [6] states that $P_{22}=P_{33}=P_{23}=0$ on Σ.

From the above, the following important observation can be made:

Consider at least one side of Σ (which coincides with a neck, say R^+) to currently be deforming plastically. If in addition, Σ coincides with one of the stress characteristic directions, say direction A (see Figure 3),

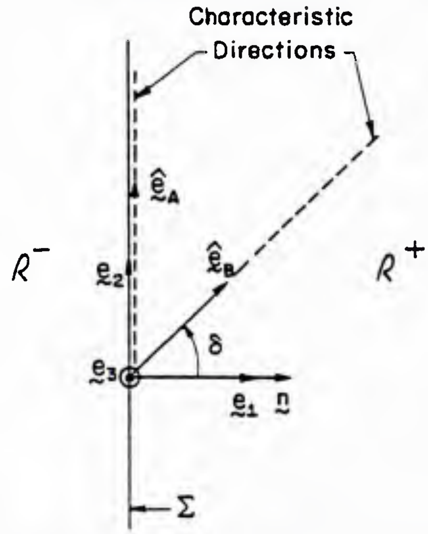

Figure 3. Moving discontinuity surface and characteristic directions of the plastically deforming side.

then the velocity component along the other characteristic direction B, is continuous across Σ. Thus

$$[V_B] = 0 \quad \text{on } \Sigma. \tag{6.14}$$

The above follows by first observing that since Σ coincides with a stress characteristic direction, P_{22} vanishes on Σ. Also, if the other characteristic direction makes an angle δ ($\delta \neq \pm \frac{\pi}{2}$) with the x_1 axis, then by the fact that $P_{BB}=0$ and the transformation relation, we have

$$\tan \delta = -\frac{P_{11}}{2P_{12}} \quad \text{for } P_{12} \neq 0. \tag{6.15}$$

In addition, combining (6.6) and (6.13) and noting that $P_{12} \neq 0$ the following is true:

$$[v_1] = \frac{P_{11}}{2P_{12}}[v_2] \quad \text{on } \Sigma. \tag{6.16}$$

The velocity jump $[v_B]$ along the other characteristic direction will be given by:

$$[v_B] = \cos\delta([v_1] + [v_2]\tan\delta), \quad \delta \neq \pm\frac{\pi}{2},$$

which vanishes by use of (6.15) and (6.16). This general result was also noted by Pan [7] for the special case of an isotropic Huber-von Mises solid and it also holds for stationary necks in a rigid-plastic solid (Hill [1]).

If in addition $P_{12}=0$, both the stress characteristics merge along Σ ($\delta = \pm \frac{\pi}{2}$) and as a result Σ becomes a "parabolic line." Equations (6.6) and (6.13) then imply that if $P_{12}=0$,

$$[v_2] = 0 \quad \text{on } \Sigma. \tag{6.17}$$

Thus when Σ coincides with a "parabolic line," the tangential velocity is continuous and *only* the normal velocity has a jump.

When Σ coincides with a neck and the two characteristic directions do not merge along Σ (see Fig. 3) then the accumulation of plastic work (6.7) due to the passage of Σ becomes:

$$W^P = \frac{1}{V}\left(\frac{\sigma_{11}P_{11} + 2\sigma_{12}P_{12}}{2P_{12}}\right)[v_2] \geq 0 \tag{6.18}$$

Also the fact that $\sigma_{ij}\dot{\varepsilon}_{ij}^P \geq 0$ implies that

$$\sigma_{ij} P_{ij} \geq 0 \tag{6.19}$$

By (2.2) and $P_{22}=0$ along Σ, (6.19) becomes:

$$\sigma_{11} P_{11} + 2\sigma_{12} P_{12} \geq 0 \qquad (6.20)$$

Inequalities (6.18) and (6.20) result to

$$\frac{1}{V} \frac{[v_2]}{2P_{12}} \geq 0, \quad P_{12} \neq 0 \qquad (6.21)$$

when the two characteristics merge along Σ ($\delta = \pm \frac{\pi}{2}$ and $P_{22}=P_{12}=0$) it follows from (6.7) and (6.17) that

$$\frac{\sigma_{11}}{V} [v_1] \geq 0. \qquad (6.22)$$

Isotropic Huber-von Mises solids

The above results can now be specialized for an isotropic elastic-perfectly plastic solid which obeys the Huber-von Mises yield condition. For such a solid, the yield condition states

$$f(\underset{\sim}{\sigma}) = \frac{1}{2} \underset{\sim}{S} \cdot \underset{\sim}{S} - \tau_o^2 = 0 \quad \text{on R} \qquad (6.23)$$

where $\underset{\sim}{S} \equiv \underset{\sim}{\sigma} - \frac{1}{3} \mathrm{tr} \underset{\sim}{\sigma} \underset{\sim}{1}$ is the deviatoric stress tensor and τ_o is the yield stress in pure shear. For such a solid

$$\underset{\sim}{P}(\underset{\sim}{\sigma}) = \nabla_{\underset{\sim}{\sigma}} f(\underset{\sim}{\sigma}) = \underset{\sim}{S} \quad \text{on R} \qquad (6.24)$$

All the results and corresponding remarks from (6.3)-(6.22) hold for this solid with $\underset{\sim}{P}$ replaced by $\underset{\sim}{S}$. In particular (6.16) takes the form (Pan [7]):

$$[v_1] = \frac{S_{11}}{2S_{12}} [v_2] \quad \text{if } S_{12} \neq 0 \qquad (6.25)$$

and (6.18) reduces to

$$W^P = \frac{1}{V} \left(\frac{\tau_o^2}{\sigma_{12}} \right) [v_2] \geq 0$$

(6.26)

Summary of Results

The results of Section 6 can now be summarized as follows:

a. For a general anisotropic hardening solid which is also plastically incompressible the following is true: When the propagating surface Σ does *not* coincide with a neck (full displacement continuity), only a jump in the tangential velocity component (sliding discontinuity) is admissible.

b. If, however, the solid is perfectly plastic, Σ coincides with one characteristic direction (P_{22}=0). In addition, full displacement continuity together with plastic incompressibility also give $P_{33}=P_{11}=0$. This states that the direction normal to Σ is *also* a characteristic direction. Unlike plane strain, this occurs under plane stress condition only under exceptional circumstances (Hill [3]). In particular, for Huber-von Mises solids this is true when the surface coincides with a plane of maximum shear stress and the latter is equal in magnitude to the yield stress in pure shear.

c. For a general anisotropic elastic-perfectly plastic solid, when a surface *coincides* with a neck (discontinuity in u_3) both tangential and normal velocities have jumps. This requires that the neck should lie along one characteristic direction. Then the component of the velocity along the other characteristic direction (not generally perpendicular to Σ) is continuous (see equation (6.14)). Thus necks cannot form if the plastically deforming side of the surface is in an elliptic state of stress.

d. For an elastic perfectly plastic solid if in addition to (c), P_{12}=0, both the characteristics merge along the neck and this results in a parabolic stress state. Then, the tangential velocity is continuous and *only the normal velo-*

city has a jump. For the special case of a Huber-von Mises solid, $P_{12}=S_{12}=0$ and the characteristic surface coincides with a principal stress direction.

7. REMARKS AND APPLICATIONS

The jump conditions discussed here have some relevance to the stress and strain fields near the tip of a quasistatically growing crack in an elastic-plastic solid under generalized plane stress conditions. For instance, in the elastic-perfectly plastic Huber-von Mises material (Rice [9]) a "constant stress" (asymptotic) plastic sector cannot occur directly behind a "centered fan" plastic sector because the condition for positive plastic work accumulation (6.21) will be violated at the interface. This renders the asymptotic solution for the plane stress stationary crack by Hutchinson [10] unacceptable when the crack begins to grow. From the preliminary asymptotic analysis by Rice [9], it then follows that only an "elastic unloading" sector can occur behind the centered fan. Hutchinson's stationary crack solution also has a jump in the in-plane stress component between two constant stress sectors. This is also inadmissible when the crack begins to propagate.

No solution for this problem which satisfies all the conditions set forth in the present paper has yet been constructed. An open question that arises, for which detailed experimental and numerical studies may provide an answer, is whether necking occurs near the growing crack tip. Otherwise, except in special circumstances (like a fan angle of 90°), no strong discontinuities near the growing crack tip can be admitted. In view of the fact that the (fully yielded) stationary crack tip solution [10] has a strong discontinuity, one wonders whether the condition of full continuity in both stress and velocity near the propagating crack tip may be too restrictive to satisfy.

8. CONCLUSION

Strong discontinuities across quasistatically propagating surfaces in anisotropic elastic-plastic solids under generalized plane stress have been reexamined allowing for some generality in constitutive response and taking into account the phenomenon of necking. Jumps in stresses have been ruled out on the basis of material stability postulates and a previous approach (by Pan [7]) has been discussed. It has been noted that for elastic-perfectly plastic solids, sliding velocity discontinuities occur under restrictive and exceptional conditions (when both the surface and its normal are stress characteristics) for generalized plane stress as compared to plane strain. Necks may form along (stress) characteristic directions with the relative velocity vector orthogonal to the other family of characteristics.

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