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A POINT LOAD IN THE INTERIOR OF A THICK PLATE

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Abstract—The problem considered here is that of the application of a constant force to an infinite three-dimensional, linear, elastic, isotropic, homogeneous layer. This force is assumed to be concentrated at any internal arbitrary point. The method of solution is based on the use of integral transforms. The distributions of stresses and displacements at all points of the solid are calculated in terms of convergent semi-infinite integrals.

The exact solution of the stress and displacement fields for an infinite plate is obtained when the concentrated point load is either close to or distant from the observed point. In the first case the Kelvin state solution of the problem of a concentrated load at a point of an elastic medium occupying the entire space is recovered. In the second case, the solution of an infinite thin plate is reproduced.

In Sec. 3 the governing partial differential field equations, defined in Sec. 2, are reduced to a system of ordinary differential equations by the use of the two-dimensional Fourier transform, taken with respect to the two in-plane geometric variables. Analytical expressions for the stresses and displacements are then obtained for the particular case of concentrated body forces, represented as Dirac delta functions (Sec. 5).

Representative stress and displacement components are plotted in the final section of the paper.

1. INTRODUCTION

The problem of a single elastic layer in equilibrium was first considered by Dougall in 1904 [1], who conducted an extensive study of a thick plate subjected to arbitrary (surface or internal) loading using potential functions. Teodone [2] also dealt with this problem by using the method of mapping. Later, Orlando [3] obtained the solution of the layer under surface tractions.

In addition to these works, Lur'e [4, 5] proposed a method to construct particular solutions of the equations of elasticity for a layer subjected to surface loads.

Marguerre's paper [6] contains numerical results of the solution of the problem of a layer compressed by concentrated forces. Also, Shapiro [7] and Sneddon [8] analyzed the distribution of stresses in an infinite layer for the case of normal loading, uniformly distributed over the area of a circle on the surface; this last author also evaluated the stress field under an approximation assumption which allowed him to obtain a closed-form expression for the semi-infinite integrals presented in his work.

The object of the present paper is to apply the transfer matrix formulation, used by Vlasov and Leont'ev [9] and generalized by Bufler [10], to the problem of a three-dimensional layer containing an internal concentrated unit load, which may act perpendicularly or parallel to the faces of the solid.

As an initial hypothesis no approximation is assumed except small deformation.

2. GENERAL EQUATIONS

The balance law and constitutive equations of a homogeneous, isotropic linear elastic body are:

$$\nabla\cdot\boldsymbol{\sigma}(\mathbf{x})+\mathbf{F}(\mathbf{x})=0$$
$$\boldsymbol{\sigma}(\mathbf{x})=\boldsymbol{\sigma}^T(\mathbf{x}) \tag{1}$$

$$\boldsymbol{\sigma}(\mathbf{x})=\frac{\nu E}{(1+\nu)(1-2\nu)}[\nabla\cdot\mathbf{u}(\mathbf{x})]\mathbf{1}+\frac{E}{2(1+\nu)}[\nabla\mathbf{u}(\mathbf{x})+\nabla^T\mathbf{u}(\mathbf{x})], \tag{2}$$

where E and ν are Young's modulus and Poisson's ratio respectively, and $\boldsymbol{\sigma}$, $\mathbf{u}$ and $\mathbf{F}$ are defined on a spatial region R.

Suitable combination of the previous expressions, in component form, can be presented in the form of a matrix differential equation [10–12]:

$$\frac{\partial\mathbf{a}}{\partial z}=\mathbf{A}\mathbf{a}+\mathbf{C} \tag{3}$$

$$\mathbf{b}=\mathbf{B}\mathbf{a}, \tag{4}$$

where $\mathbf{a}$ and $\mathbf{b}$ define the column vectors

$$\mathbf{a}=(\sigma_{zz},\sigma_{zx},\sigma_{zy},u_y,u_x,u_z)^T \tag{5}$$

$$\mathbf{b}=(\sigma_{xx}+\sigma_{yy},\sigma_{xx}-\sigma_{yy},2\sigma_{xy})^T; \tag{6}$$

x, y, z are spatial Cartesian coordinates in a Euclidean 3-space, $(\ )^T$ stands for the transpose of a vector and matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 & 0 & 0 \\ -\frac{\nu}{1-\nu}\frac{\partial}{\partial x} & 0 & 0 & -\frac{E}{2(1-\nu)}\frac{\partial^2}{\partial x\,\partial y} & \left(-\frac{E}{2(1+\nu)}\frac{\partial^2}{\partial y^2}-\frac{E}{1-\nu^2}\frac{\partial^2}{\partial x^2}\right) & 0 \\ -\frac{\nu}{1-\nu}\frac{\partial}{\partial y} & 0 & 0 & \left(-\frac{E}{2(1+\nu)}\frac{\partial^2}{\partial x^2}-\frac{E}{1-\nu^2}\frac{\partial^2}{\partial y^2}\right) & -\frac{E}{2(1-\nu)}\frac{\partial^2}{\partial x\,\partial y} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 & -\frac{\partial}{\partial y} \\ 0 & \frac{2(1+\nu)}{E} & 0 & 0 & 0 & -\frac{\partial}{\partial x} \\ \frac{(1+\nu)(1-2\nu)}{(1-\nu)E} & 0 & 0 & -\frac{\nu}{1-\nu}\frac{\partial}{\partial y} & -\frac{\nu}{1-\nu}\frac{\partial}{\partial x} & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \frac{2\nu}{1-\nu} & 0 & 0 & \frac{E}{1-\nu}\frac{\partial}{\partial y} & \frac{E}{1-\nu}\frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & -\frac{E}{1+\nu}\frac{\partial}{\partial y} & \frac{E}{1+\nu}\frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & \frac{E}{1+\nu}\frac{\partial}{\partial x} & \frac{E}{1+\nu}\frac{\partial}{\partial y} & 0 \end{bmatrix}.$$

The column vector $\mathbf{C}$ is defined as

$$\mathbf{C} = (-F_z, -F_x, -F_y, 0, 0, 0)^T,$$

where F_x, F_y, F_z stand for the body force components.

3. INFINITE LAYER: TRANSFORMED GENERAL EQUATIONS

The matrix differential eqn (3) relates to the z-coordinate partial derivative of vector $\mathbf{a}$ with the vector $\mathbf{a}$ itself. The vector $\mathbf{a}$ is composed of the components of the tractions acting on a constant-z plane as well as the components of the displacements.

If x, y are the in-plane coordinates of the layer and z is the coordinate perpendicular to the faces, the matrix partial differential eqn (3) can be transformed into an ordinary matrix differential equation by using the two-dimensional Fourier transform with respect to the coordinates x, y.

According to Sneddon [13] and Bufler [10], the following geometric Fourier transforms are defined:

$$\bar{f}(\alpha,\beta) = \mathscr{F}[f(x,y)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,\mathrm{e}^{i(\alpha x+\beta y)}\,\mathrm{d}x\,\mathrm{d}y$$

$$\overset{\alpha}{\bar{f}}(\alpha,\beta) = \mathscr{F}_\alpha[f(x,y)] = \frac{1}{j_\alpha}\mathscr{F}[f(x,y)]$$

$$\overset{\beta}{\bar{f}}(\alpha,\beta) = \mathscr{F}_\beta[f(x,y)] = \frac{1}{j_\beta}\mathscr{F}[f(x,y)],$$

where

$$j_\alpha = i\frac{\alpha}{|\alpha|} \quad (\alpha \neq 0)$$

$$j_\alpha = i \quad (\alpha = 0)$$

$$j_\beta = i\frac{\beta}{|\beta|} \quad (\beta \neq 0)$$

$$j_\beta = i \quad (\beta = 0)$$

$$i = \sqrt{-1}.$$

With inverse transform of the form

$$f(x,y) = \begin{pmatrix} \mathscr{F}^{-1}[\bar{f}(\alpha,\beta)] \\ \mathscr{F}_\alpha^{-1}[\bar{f}^\alpha(\alpha,\beta)] \\ \mathscr{F}_\beta^{-1}[\bar{f}^\beta(\alpha,\beta)] \end{pmatrix} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \begin{pmatrix} \bar{f}(\alpha,\beta) \\ j_\alpha\bar{f}^\alpha(\alpha,\beta) \\ j_\beta\bar{f}^\beta(a,\beta) \end{pmatrix} \times \mathrm{e}^{-i(\alpha x+\beta y)}\,\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{7}$$

and with derivatives of the form

$$\left\{ \begin{aligned} &\overline{\frac{\partial f}{\partial x}} = |\alpha|\overset{\alpha}{\bar{f}} = -j_\alpha j_\beta|\alpha|\overset{\beta}{\bar{f}} \\ &\overset{\alpha}{\overline{\frac{\partial f}{\partial x}}} = -|\alpha|\bar{f} \\ &\overset{\alpha}{\overline{\frac{\partial^2 f}{\partial x^2}}} = -\alpha^2\overset{\alpha}{\bar{f}} \\ &\overset{\beta}{\overline{\frac{\partial^2 f}{\partial x^2}}} = -\alpha^2\overset{\beta}{\bar{f}} \\ &\overset{\alpha}{\overline{\frac{\partial^2 f}{\partial x\,\partial y}}} = -|\alpha||\beta|\overset{\beta}{\bar{f}} \end{aligned} \right\}, \tag{8}$$

the function f is such that the following conditions are fulfilled:

$$\left\{\begin{aligned} &f(x,y)\to 0 \quad \text{as } |x|\to\infty \text{ and/or } |y|\to\infty \\ &\frac{\partial f(x,y)}{\partial x}\to 0 \quad \text{as } |x|\to\infty \\ &\frac{\partial f(x,y)}{\partial y}\to 0 \quad \text{as } |y|\to\infty \end{aligned}\right\}.$$

By the application of the matrix operators

$$\mathbf{F}=\begin{bmatrix} \mathscr{F} & & & & & \\ & \mathscr{F}_\alpha & & & & \\ & & \mathscr{F}_\beta & & & \\ & & & \mathscr{F}_\beta & & \\ & & & & \mathscr{F}_\alpha & \\ & & & & & -\mathscr{F} \end{bmatrix}$$

$$\mathbf{F}'=\begin{pmatrix} \mathscr{F} & & \\ & -\mathscr{F} & \\ & & -\mathscr{F} \end{pmatrix}$$

to eqns (3) and (4), and by using (8), these expressions yield

$$\frac{\mathrm{d}\bar{\mathbf{a}}}{\mathrm{d}z}=\bar{\mathbf{A}}\bar{\mathbf{a}}+\bar{\mathbf{C}} \qquad (9)$$

$$\bar{\mathbf{b}}=\bar{\mathbf{B}}\bar{\mathbf{a}}, \qquad (10)$$

where

$$\bar{\mathbf{a}}=\mathbf{F}\mathbf{a}=(\bar{\sigma}_{zz},\overset{\alpha}{\bar{\sigma}}_{zx},\overset{\beta}{\bar{\sigma}}_{zy},\overset{\beta}{\bar{u}}_y,\overset{\alpha}{\bar{u}}_x,\bar{u}_z)^T$$

$$\bar{u}_z=\mathscr{F}[-u_z] \qquad (11a)$$

$$\bar{\mathbf{b}}=\mathbf{F}'\mathbf{a}=(\bar{\sigma}_{xx}+\bar{\sigma}_{yy},-[\bar{\sigma}_{xx}-\bar{\sigma}_{yy}],-2\bar{\sigma}_{xy})^T \qquad (11b)$$

$$\bar{\mathbf{C}}=\mathbf{F}\mathbf{C}=(-\bar{F}_z,\overset{\alpha}{\bar{F}}_x,-\overset{\beta}{\bar{F}}_y,0,0,0)^T. \qquad (11c)$$

According to Bufler[10] we shall define the dimensionless transform parameters as

$$\alpha^*=|\alpha|h$$

$$\beta^*=|\beta|h$$

$$\lambda=\sqrt{(\alpha^{*2}+\beta^{*2})}.$$

Thus, matrices $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are

$$\bar{\mathbf{A}}=\begin{bmatrix} 0 & -\frac{\alpha^*}{h} & -\frac{\beta^*}{h} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu}\frac{\alpha^*}{h} & 0 & 0 & \frac{1}{2(1-\nu)}E\frac{\alpha^*\beta^*}{h^2} & \frac{1}{1-\nu^2}E\left(\frac{2\alpha^{*2}+(1-\nu)\beta^{*2}}{2h^2}\right) & 0 \\ \frac{\nu}{1-\nu}\frac{\beta^*}{h} & 0 & 0 & \frac{1}{1-\nu^2}E\left(\frac{2\beta^{*2}+(1-\nu)\alpha^{*2}}{2h^2}\right) & \frac{1}{2(1-\nu)}E\frac{\alpha^*\beta^*}{h^2} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 & -\frac{\beta^*}{h} \\ 0 & \frac{2(1+\nu)}{E} & 0 & 0 & 0 & -\frac{\alpha^*}{h} \\ \frac{-(1+\nu)(1-2\nu)}{(1-\nu)E} & 0 & 0 & \frac{\nu}{1-\nu}\frac{\beta^*}{h} & \frac{\nu}{1-\nu}\frac{\alpha^*}{h} & 0 \end{bmatrix}$$

$$\bar{\mathbf{B}}=\begin{bmatrix} \frac{2\nu}{1-\nu} & 0 & 0 & \frac{1}{1-\nu}E\frac{\beta^*}{h} & \frac{1}{1-\nu}E\frac{\alpha^*}{h} & 0 \\ 0 & 0 & 0 & \frac{1}{1+\nu}E\frac{\beta^*}{h} & -\frac{1}{1+\nu}E\frac{\alpha^*}{h} & 0 \\ 0 & 0 & 0 & \frac{1}{1+\nu}E\frac{\alpha^*}{h}j_\alpha j_\beta & \frac{1}{1+\nu}E\frac{\beta^*}{h}j_\alpha j_\beta & 0 \end{bmatrix}.$$

The vector $\bar{\mathbf{a}}$ will be referred to as the state vector.

4. MATRIX DIFFERENTIAL EQUATION

Equation (9) is an ordinary matrix differential equation which can be solved using the Cayley–Hamilton theorem [14].

For an arbitrary point at a distance z from the lower surface of the layer, the state vector is given by

$$\bar{\mathbf{a}}(z) = \mathbf{X}(z)\mathbf{X}^{-1}(0)\bar{\mathbf{a}}(0) + \mathbf{X}(z)\int_0^z \mathbf{X}^{-1}(s)\bar{\mathbf{C}}(s)\,\mathrm{d}s, \quad (12)$$

where $\bar{\mathbf{a}}(0)$ represents the initial value of $\bar{\mathbf{a}}$ at $z = 0$, $\mathbf{X}(z)$ is the fundamental matrix defined by the matrix of eigenvectors of $\bar{\mathbf{A}}$ postmultiplied by the matrix of eigenvalues of $\bar{\mathbf{A}}$ and s is a dummy variable.

(a) *Transfer matrix*

The transfer matrix is given by $\mathbf{X}(z)\mathbf{X}^{-1}(0)$, and relates the state vector $\bar{\mathbf{a}}(z)$ of any arbitrary point z with the inital state vector $\bar{\mathbf{a}}(0)$.

We shall denote it by

$$\mathbf{T}(z) \equiv \mathbf{X}(z)\mathbf{X}^{-1}(0) = \frac{1}{2(1-\nu)}\cosh\left(\lambda\frac{z}{h}\right)[t_{ij}], \quad (13)$$

where

$$t_{11} = 2(1-\nu) - \lambda\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{12} = -\frac{\alpha^*}{\lambda}\left[\lambda\frac{z}{h} + (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{13} = -\frac{\beta^*}{\lambda}\left[\lambda\frac{z}{h} + (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{14} = -\frac{E}{h}\frac{1}{1+\nu}\lambda\beta^*\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{15} = -\frac{E}{h}\frac{1}{1+\nu}\lambda\alpha^*\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{16} = \frac{E}{h}\frac{1}{1+\nu}\lambda\left[\lambda\frac{z}{h} - \tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{21} = \frac{\alpha^*}{\lambda}\left[\lambda\frac{z}{h} - (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{22} = 2(1-\nu) + \frac{\alpha^{*2}}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{23} = \frac{\alpha^*\beta^*}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{24} = \frac{E}{h}\frac{1}{1+\nu}\frac{\alpha^*\beta^*}{\lambda}\left[\lambda\frac{z}{h} + \nu\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{25} = \frac{E}{h}\frac{1}{1+\nu}\lambda\left[\frac{\alpha^{*2}}{\lambda}\frac{z}{h} + \left(1 - \nu\frac{\beta^{*2}}{\lambda^2}\right)\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{26} = -\frac{E}{h}\frac{1}{1+\nu}\lambda\alpha^*\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{31} = \frac{\beta^*}{\lambda}\left[\lambda\frac{z}{h} - (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{32} = \frac{\alpha^*\beta^*}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{33} = 2(1-\nu) + \frac{\beta^{*2}}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{34} = \frac{E}{h}\frac{1}{1+\nu}\lambda\left[\frac{\beta^{*2}}{\lambda}\frac{z}{h} + \left(1 - \nu\frac{\alpha^{*2}}{\lambda^2}\right)\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{35} = \frac{E}{h}\frac{1}{1+\nu}\frac{\alpha^*\beta^*}{\lambda}\left[\lambda\frac{z}{h} + \nu\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{36} = -\frac{E}{h}\frac{1}{1+\nu}\lambda\beta^*\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{41} = \frac{h}{E}(1+\nu)\frac{\beta^*}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{42} = \frac{h}{E}(1+\nu)\frac{\alpha^*\beta^*}{\lambda^2}\left[\frac{z}{h} - \frac{1}{\lambda}\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{43} = \frac{h}{E}(1+\nu)\frac{1}{\lambda}\left\langle 4(1-\nu)\tanh\left(\lambda\frac{z}{h}\right) - \frac{\beta^{*2}}{\lambda^2}\left[\tanh\left(\lambda\frac{z}{h}\right) - \lambda\frac{z}{h}\right]\right\rangle$$

$$t_{44} = 2(1-\nu) + \frac{\beta^{*2}}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{45} = \frac{\alpha^*\beta^*}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{46} = -\frac{\beta^*}{\lambda}\left[\lambda\frac{z}{h} + (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{51} = \frac{h}{E}(1+\nu)\frac{\alpha^*}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{52} = \frac{h}{E}(1+\nu)\frac{1}{\lambda}\left\langle 4(1-\nu)\tanh\left(\lambda\frac{z}{h}\right) - \frac{\alpha^{*2}}{\lambda^2}\left[\tanh\left(\lambda\frac{z}{h}\right) - \lambda\frac{z}{h}\right]\right\rangle$$

$$t_{53} = \frac{h}{E}(1+\nu)\frac{\alpha^*\beta^*}{\lambda^2}\left[\frac{z}{h} - \frac{1}{\lambda}\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{54} = \frac{\alpha^*\beta^*}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{55} = 2(1-\nu) + \frac{\alpha^{*2}}{\lambda}\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{56} = -\frac{\alpha^*}{\lambda}\left[\lambda\frac{z}{h} + (1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{61} = \frac{h}{E}(1+\nu)\left[\frac{z}{h} - \frac{3-4\nu}{\lambda}\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{62}=\frac{h}{E}(1+\nu)\frac{\alpha^* z}{\lambda\, h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{63}=\frac{h}{E}(1+\nu)\frac{\beta^* z}{\lambda\, h}\tanh\left(\lambda\frac{z}{h}\right)$$

$$t_{64}=\frac{\beta^*}{\lambda}\left[\lambda\frac{z}{h}-(1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{65}=\frac{\alpha^*}{\lambda}\left[\lambda\frac{z}{h}-(1-2\nu)\tanh\left(\lambda\frac{z}{h}\right)\right]$$

$$t_{66}=2(1-\nu)-\lambda\frac{z}{h}\tanh\left(\lambda\frac{z}{h}\right).$$

(b) *Flexibility matrices*

Expression (12), using (13), can be written as

$$\bar{\mathbf{a}}(z)=\mathbf{T}(z)\bar{\mathbf{a}}(0)+\bar{\mathbf{R}}(z), \tag{14}$$

where

$$\bar{\mathbf{R}}(z)=\mathbf{X}(z)\int_0^z \mathbf{X}^{-1}(s)\bar{\mathbf{C}}(s)\,\mathrm{d}s.$$

$$\begin{pmatrix}\bar{\boldsymbol{\sigma}}(z)\\ \bar{\mathbf{u}}(z)\end{pmatrix}=\begin{pmatrix}\mathbf{T}_{11}(z)-\mathbf{T}_{12}(z)\mathbf{T}_{12}^{-1}(h)\mathbf{T}_{11}(h) & \mathbf{T}_{12}(z)\mathbf{T}_{12}^{-1}(h)\\ \mathbf{T}_{21}(z)-\mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(h)\mathbf{T}_{11}(h) & \mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(h)\end{pmatrix}\begin{pmatrix}\bar{\boldsymbol{\sigma}}(0)\\ \bar{\boldsymbol{\sigma}}(h)\end{pmatrix}+\begin{pmatrix}-\mathbf{T}_{12}(z)\mathbf{T}_{12}^{-1}(h)\bar{\mathbf{R}}_1(h)+\bar{\mathbf{R}}_1(z)\\ -\mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(h)\bar{\mathbf{R}}_1(h)+\bar{\mathbf{R}}_2(z)\end{pmatrix}, \tag{17}$$

The state vector $\bar{\mathbf{a}}$ is composed of stresses and displacements in the following way:

$$\bar{\mathbf{a}}(z)=\begin{pmatrix}\bar{\boldsymbol{\sigma}}(z)\\ \bar{\mathbf{u}}(z)\end{pmatrix},$$

where

$$\bar{\boldsymbol{\sigma}}(z)=(\bar{\sigma}_{zz},\overset{\alpha}{\bar{\sigma}}_{xz},\overset{\beta}{\bar{\sigma}}_{zy})^T,\quad \bar{\mathbf{u}}(z)=(\overset{\beta}{\bar{u}}_y,\overset{\alpha}{\bar{u}}_x,\bar{u}_z)^T.$$

If we are interested in relating displacements with stresses, e.g. stresses are known by the boundary conditions of our problem, this can be done using expression (14) in the form

$$\begin{pmatrix}\bar{\boldsymbol{\sigma}}(z)\\ \bar{\mathbf{u}}(z)\end{pmatrix}=\begin{pmatrix}\mathbf{T}_{11}(z) & \mathbf{T}_{12}(z)\\ \mathbf{T}_{21}(z) & \mathbf{T}_{22}(z)\end{pmatrix}\begin{pmatrix}\bar{\boldsymbol{\sigma}}(0)\\ \bar{\mathbf{u}}(0)\end{pmatrix}+\begin{pmatrix}\bar{\mathbf{R}}_1(z)\\ \mathbf{R}_2(z)\end{pmatrix},$$

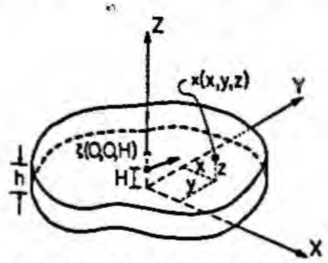

Fig. 1. Single layer with unit internal load applied to an arbitrary point. Point X is an arbitrary internal point.

where $\mathbf{T}_{ij}(z)$ $(\forall i,j=1,2)$ stand for the submatrices of $\mathbf{T}(z)$ defined in (13) and $\bar{\mathbf{R}}_i(z)$ are two column vectors containing the first three and the last three components of $\bar{\mathbf{R}}(z)$ respectively.

From the former expression

$$\begin{pmatrix}\bar{\mathbf{u}}(0)\\ \bar{\mathbf{u}}(z)\end{pmatrix}=\begin{pmatrix}-\mathbf{T}_{12}^{-1}(z)\mathbf{T}_{11}(z) & \mathbf{T}_{12}^{-1}(z)\\ \mathbf{T}_{21}(z)-\mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(z)\mathbf{T}_{11}(z) & \mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(z)\end{pmatrix}\times\begin{pmatrix}\bar{\boldsymbol{\sigma}}(0)\\ \bar{\boldsymbol{\sigma}}(z)\end{pmatrix}+\begin{pmatrix}-\mathbf{T}_{12}^{-1}(z) & 0\\ -\mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(z) & 1\end{pmatrix}\begin{pmatrix}\bar{\mathbf{R}}_1(z)\\ \bar{\mathbf{R}}_2(z)\end{pmatrix}. \tag{15}$$

(c) *Internal point state vector*

For an arbitrary point, inside the layer, eqn (14) holds:

$$\bar{\mathbf{a}}(z)=\mathbf{T}(z)\bar{\mathbf{a}}(0)+\bar{\mathbf{R}}(z). \tag{16}$$

Also, for $z=h$, from (15) we obtain

$$\bar{\mathbf{u}}(0)=-\mathbf{T}_{12}^{-1}(h)\mathbf{T}_{11}(h)\bar{\boldsymbol{\sigma}}(0)+\mathbf{T}_{12}^{-1}(h)\bar{\boldsymbol{\sigma}}(h)-\mathbf{T}_{12}^{-1}(h)\bar{\mathbf{R}}_1(h).$$

Substituting $\bar{\mathbf{u}}(0)$ from above into eqn (16), we obtain

which represents the transforms of the stresses and displacements of any arbitrary point with respect to the transforms of the tractions on the surfaces and the transform of the applied body forces.

5. INFINITE LAYER WITH CONCENTRATED BODY FORCES

In this section we shall consider the solution of a layer subjected to concentrated forces of unit magnitude acting in an arbitrary direction and applied to any internal point.

Let $\xi(0,0,H)$ be the point where the force is applied and $\mathbf{x}(x,y,z)$ be the point of observation, as depicted in Fig. 1. If $\delta(x,y,z)$ stands for the Dirac delta function defined in the geometric domain, arbitrary forces in the three directions will be expressed as

$$F^x=(\delta(x,y,z-H)\mathbf{e}_x,0,0)^T$$

$$F^y=(0,\delta(x,y,z-H)\mathbf{e}_y,0)^T$$

$$F^z=(0,0,\delta(x,y,z-H)\mathbf{e}_z)^T.$$

The transformed expressions for the body forces are given by

$$\overset{\alpha}{\bar{F}}{}^x = \frac{1}{2\pi}\frac{1}{j_\alpha}\,\delta(z-H)\mathbf{e}_x$$

$$\overset{\beta}{\bar{F}}{}^y = \frac{1}{2\pi}\frac{1}{j_\beta}\,\delta(z-H)\mathbf{e}_y$$

$$\bar{F}^z = \frac{1}{2\pi}\,\delta(z-H)\mathbf{e}_z, \tag{18}$$

where $\mathbf{e}_i$ is a unit vector in the ith direction.

By applying, sequentially, expressions (18) to the second term of the right hand side of (12) or (14), we obtain

$$\bar{R}^x(z) = \mathbf{X}(z)\mathbf{X}^{-1}(H)\left(0, -\frac{1}{2\pi}\frac{1}{j_\alpha}, 0, 0, 0, 0\right)^T$$

$$\bar{R}^y(z) = \mathbf{X}(z)\mathbf{X}^{-1}(H)\left(0, 0, -\frac{1}{2\pi}\frac{1}{j_\beta}, 0, 0, 0\right)^T$$

$$\bar{R}^z(z) = \mathbf{X}(z)\mathbf{X}^{-1}(H)\left(-\frac{1}{2\pi}, 0, 0, 0, 0, 0\right)^T,$$

or

$$\bar{R}^x(z) = \frac{1}{4\pi(1-\nu)\lambda j_\alpha}\left\{\begin{array}{l}
\alpha^*\left[(1-2\nu)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] + \lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right] \\
-\left[\alpha^{*2}\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] + 2(1-\nu)\lambda\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right] \\
-\alpha^*\beta^*\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] \\
\frac{h}{E}(1+\nu)\frac{\alpha^*\beta^*}{\lambda^2}\left[\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] - \lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right] \\
\frac{h}{E}(1+\nu)\frac{1}{\lambda^2}\left[(\alpha^{*2}-4(1-\nu)\lambda^2)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] - \alpha^{*2}\lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right] \\
-\frac{h}{E}(1+\nu)\alpha^*\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]
\end{array}\right\} \tag{19a}$$

$$\bar{R}^y(z) = \frac{1}{4\pi(1-\nu)\lambda j_\beta}\left\{\begin{array}{l}
\beta^*\left[(1-2\nu)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] + \lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right] \\
-\alpha^*\beta^*\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] \\
-\left[\beta^{*2}\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] + 2(1-\nu)\lambda\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right] \\
\frac{h}{E}(1+\nu)\frac{1}{\lambda^2}\left[(\beta^{*2}-4(1-\nu)\lambda^2)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] - \beta^{*2}\lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right] \\
\frac{h}{E}(1+\nu)\frac{\alpha^*\beta^*}{\lambda^2}\left[\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] - \lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right] \\
-\frac{h}{E}(1+\nu)\beta^*\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right]
\end{array}\right\} \tag{19b}$$

$$\bar{R}^z(z) = \frac{1}{4\pi(1-\nu)}\left\{\begin{array}{l}
\lambda\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] - 2(1-\nu)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right] \\
\frac{\alpha^*}{\lambda}\left[(1-2\nu)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] - \lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right] \\
\frac{\beta^*}{\lambda}\left[(1-2\nu)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] - \lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right] \\
-\frac{h}{E}(1+\nu)\frac{\beta^*}{\lambda}\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] \\
-\frac{h}{E}(1+\nu)\frac{\alpha^*}{\lambda}\left(\frac{z-H}{h}\right)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] \\
-\frac{h}{E}(1+\nu)\frac{1}{\lambda}\left[(4\nu-3)\sinh\left[\lambda\left(\frac{z-H}{h}\right)\right] + \lambda\left(\frac{z-H}{h}\right)\cosh\left[\lambda\left(\frac{z-H}{h}\right)\right]\right]
\end{array}\right\} \tag{19c}$$

where the superscript (x, y, z) denotes the direction of the unit load.

(a) *Internal point state vector*

In this particular case of concentrated body forces, we shall write

$$\bar{\sigma}(0)=\bar{\sigma}(h)=0.$$

It follows from (17) that

$$\begin{pmatrix}\bar{\sigma}(z)\\ \bar{\mathbf{u}}(z)\end{pmatrix}=\begin{pmatrix}-\mathbf{T}_{12}(z)\mathbf{T}_{12}^{-1}(h)\bar{\mathbf{R}}_1(h)+\bar{\mathbf{R}}_1(z)\\ -\mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(h)\bar{\mathbf{R}}_1(h)+\bar{\mathbf{R}}_2(z)\end{pmatrix},$$

for any point z above the load level H $(z \geqslant H)$, otherwise the additional terms $\bar{\mathbf{R}}_1(z)$ and $\bar{\mathbf{R}}_2(z)$ should be dropped.

Thus, the former expression, for any arbitrary point z, gives

$$\begin{pmatrix}\bar{\sigma}(z)\\ \bar{\mathbf{u}}(z)\end{pmatrix}$$
$$=\begin{pmatrix}-\mathbf{T}_{12}(z)\mathbf{T}_{12}^{-1}(h)\bar{\mathbf{R}}_1(h)+\bar{\mathbf{R}}_1(z)\mathscr{H}(z-H)\\ -\mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(h)\bar{\mathbf{R}}_1(h)+\bar{\mathbf{R}}_2(z)\mathscr{H}(z-H)\end{pmatrix}, \quad (20)$$

where

$$\mathscr{H}(z-H)=\begin{cases}1, & \forall z \geqslant H\\ 0, & \forall z < H\end{cases}.$$

(b) *Analytical expressions for the stresses and displacements*

From eqns (7), (20) and Appendix A, the expressions for the stresses and displacements defined in (5) can be inferred. Also, making use of the transformed stresses and displacements given by (20) and by means of eqn (10), after performing the inverse transforms in connection with (7) and by using Appendix A, the rest of the stress components will be obtained.

The analytical expressions obtained in that way are given in terms of infinite integrals. Close inspection of the expressions for the stresses $\sigma_{zz}, \sigma_{zx}, \sigma_{zy}, \sigma_{xy}$ (when the unit load is applied along the x, y or z directions), or σ_{xx} and σ_{yy} (when the unit load is applied along the x or y directions), demonstrates that the resulting integrands exist and are well behaved for every $\lambda \in [0, \infty)$. For $\lambda \to 0$, this was shown by expanding the integrands in ascending powers of λ and proving that the resulting expressions vanish as $\lambda \to 0$. For $\lambda \to \infty$, this was shown by replacing the hyperbolic functions involved by their equivalent exponential forms and demonstrating that the limit of the resulting expression, as $\lambda \to \infty$, vanishes.

On the other hand, analysis of the equivalent expressions for the displacements and for the stresses σ_{zz} and σ_{yy} (when the unit load is applied along the z direction), showed that although the integrands involved were well behaved as $\lambda \to \infty$, they became singular as $\lambda \to 0$. In fact, expansion of these integrands in ascending powers of λ revealed terms of the form:

$$A\left(\chi,\psi,\frac{R}{h}\right)\lambda^{-3}+B\left(\chi,\psi,\frac{R}{h}\right)\lambda^{-1}, \quad \lambda \to 0, \quad (21)$$

where A and B were known functions of

$$\chi=\frac{z}{h}, \quad \psi=\frac{H}{h} \quad \text{and} \quad \frac{R}{h}=\frac{\sqrt{(x^2+y^2)}}{h}$$

respectively.

The singular behaviour of the integrands indicates that the resulting expressions for the displacements are non-convergent and that the above solution should be critically re-examined.

(c) *Proposed modifications*

The construction of the final solution to our problem was suggested by the observation that simple subtraction of terms of the form

$$A\left(\chi,\psi,\frac{R}{h}\right)\lambda^{-3}+B\left(\chi,\psi,\frac{R}{h}\right)\lambda^{-1}\mathrm{e}^{-\lambda} \quad (22)$$

from the original integrands resulted in integrals for the displacements and stresses (σ^z_{xx} and σ^z_{yy}), which were convergent. It should be noted here that expression (22) reduces to (21) as $\lambda \to 0$. The inclusion of the multiplying factor $\mathrm{e}^{-\lambda}$ in the λ^{-1} term of (21) ensures the integrability of the final expressions for the displacements and stresses.

It was further observed that the functions $A\lambda^{-3}+B\lambda^{-1}\mathrm{e}^{-\lambda}$ of the transform variable $\lambda=\sqrt{(\alpha^{*2}+\beta^{*2})}$ represent Fourier transforms of displacements contributing nothing to the transformed stresses $\bar{\sigma}_{zz}, \bar{\sigma}^{\alpha}_{zx}, \bar{\sigma}^{\beta}_{zy}$, thus automatically satisfying the zero traction boundary conditions at $z=h$ and $z=0$. This was also consistent with the fact that the kernels of the integral expressions for the $\sigma_{zz}, \sigma_{zx}, \sigma_{zy}$ stresses do not involve singular terms as $\lambda \to 0$.

In addition it was shown that the stress–displacement state vectors $(\bar{\sigma}_{zz}, \bar{\sigma}^{\alpha}_{zx}, \bar{\sigma}^{\beta}_{zy}, \bar{u}^{\beta}_{y}, \bar{u}^{\alpha}_{x}, \bar{u}_z)^T$ corresponding to the transformed displacement $A\lambda^{-3}+B\lambda^{-1}\mathrm{e}^{-\lambda}$ are also solutions of the transformed governing eqns (9).

Motivated by the above observations, we propose here a solution constructed by simply subtracting singular functions of the form (22) from the integrands of the displacements. As mentioned above, the resulting displacements are convergent and give rise to stresses σ_{zz}, σ_{zx}, σ_{zy}, σ_{xx}, σ_{yy}, σ_{xy}, which are identical to the ones obtained directly from eqns (10) and (20).

In the next section we will present all displacement and stress components resulting from the modified solution. We shall then formally prove that the proposed fields satisfy all field equations, boundary conditions and reduce to the well known solution for a point load in an infinite domain (Kelvin state), as the point of application of the load is approached.

6. STRESS AND DISPLACEMENT FIELDS

$$\sigma^z_{zz}(x,y,z)=\frac{1}{4\pi(1-\nu)h^2}\int_{\lambda=0}^{\lambda=\infty} f^z_{zz}(\lambda) \times J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda \tag{23}$$

$$\sigma^z_{zx}(x,y,z)=\frac{1}{4\pi(1-\nu)h^2}\frac{x}{\sqrt{(x^2+y^2)}}\int_{\lambda=0}^{\lambda=\infty} f^z_{zx}(\lambda) \times J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda \tag{24}$$

$$\sigma^z_{zy}(x,y,z)=\frac{1}{4\pi(1-\nu)h^2}\frac{y}{\sqrt{(x^2+y^2)}}\int_{\lambda=0}^{\lambda=\infty} f^z_{zy}(\lambda) \times J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda \tag{25}$$

$$\sigma^x_{zz}(x,y,z)=\frac{1}{4\pi(1-\nu)h^2}\frac{-x}{\sqrt{(x^2+y^2)}}\int_{\lambda=0}^{\lambda=\infty} f^x_{zz}(\lambda) \times J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda \tag{26}$$

$$\begin{aligned}\sigma^x_{zx}(x,y,z)=\frac{1}{4\pi(1-\nu)h^2}\Bigg\{&\int_{\lambda=0}^{\lambda=\infty} f^x_{zx_1}(\lambda) \times J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\\ &+\frac{1}{2}\Bigg[\int_{\lambda=0}^{\lambda=\infty}(f^x_{zx_2}(\lambda)+f^x_{zx_3}(\lambda)) \times J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda+\frac{y^2-x^2}{x^2+y^2}\\ &\times\int_{\lambda=0}^{\lambda=\infty}(f^x_{zx_2}(\lambda)-f^x_{zx_3}(\lambda)) \times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\Bigg]\Bigg\}\end{aligned} \tag{27}$$

$$\sigma^x_{zy}(x,y,z)=\frac{1}{4\pi(1-\nu)h^2}\frac{-xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty} f^x_{zy}(\lambda) \times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda \tag{28}$$

$$\sigma^y_{zz}(x,y,z)=\frac{1}{4\pi(1-\nu)h^2}\frac{-y}{\sqrt{(x^2+y^2)}}\int_{\lambda=0}^{\lambda=\infty} f^y_{zz}(\lambda) \times J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda \tag{29}$$

$$\sigma^y_{zx}(x,y,z)=\frac{1}{4\pi(1-\nu)h^2}\frac{-xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty} f^x_{zy}(\lambda) \times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda \tag{30}$$

$$\begin{aligned}\sigma^y_{zy}(x,y,z)=\frac{1}{4\pi(1-\nu)h^2}\Bigg\{&\int_{\lambda=0}^{\lambda=\infty} f^y_{zy_1}(\lambda) \times J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\\ &+\frac{1}{2}\Bigg[\int_{\lambda=0}^{\lambda=\infty}(f^y_{zy_2}(\lambda)+f^y_{zy_3}(\lambda)) \times J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda-\frac{y^2-x^2}{x^2+y^2}\\ &\times\int_{\lambda=0}^{\lambda=\infty}(f^y_{zy_2}(\lambda)-f^y_{zy_3}(\lambda)) \times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\Bigg]\Bigg\}\end{aligned} \tag{31}$$

$$\begin{aligned}\sigma^z_{xx}(x,y,z)=\frac{\nu}{1-\nu}\sigma^z_{zz}(x,y,z)+\frac{1}{8\pi(1-\nu)^2h^2}&\times\Bigg\{(1+\nu)\int_{\lambda=0}^{\lambda=\infty}\lambda\Bigg[f^z_x(\lambda) \times J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\\ &-12(1-\nu)^2(2\chi-1)\frac{\mathrm{e}^{-\lambda}}{\lambda}\Bigg]\mathrm{d}\lambda\\ &+(1-\nu)\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda f^z_x(\lambda) \times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\Bigg\}\end{aligned} \tag{32}$$

$$\begin{aligned}\sigma^z_{yy}(x,y,z)=\frac{\nu}{1-\nu}\sigma^z_{zz}(x,y,z)+\frac{1}{8\pi(1-\nu)^2h^2}&\times\Bigg\{(1+\nu)\int_{\lambda=0}^{\lambda=\infty}\lambda\Bigg[f^z_y(\lambda) \times J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\\ &-12(1-\nu)^2(2\chi-1)\frac{\mathrm{e}^{-\lambda}}{\lambda}\Bigg]\mathrm{d}\lambda\\ &+(1-\nu)\frac{x^2-y^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda f^z_y(\lambda) \times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\Bigg\}\end{aligned} \tag{33}$$

$$\sigma_{xy}^{z}(x,y,z)=-\frac{1}{4\pi(1-\nu)h^2}\frac{xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda f_y^z(\lambda)\times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda \qquad (34)$$

$$\begin{aligned}\sigma_{xx}^{x}(x,y,z)&=\frac{\nu}{1-\nu}\sigma_{zz}^{x}(x,y,z)\\&+\frac{1}{16\pi(1-\nu)^2h^2}\frac{x}{\sqrt{(x^2+y^2)}}\\&\times\left\{(1-\nu)\frac{x^2-3y^2}{x^2+y^2}\right.\\&\times\int_{\lambda=0}^{\lambda=\infty}\lambda[f_{x1}^{x}(\lambda)-f_{x2}^{x}(\lambda)]\\&\times J_3\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\\&-\int_{\lambda=0}^{\lambda=\infty}\lambda[(3+\nu)f_{x1}^{x}(\lambda)\\&+(1-\nu)f_{x2}^{x}(\lambda)+4f_{x3}^{x}(\lambda)]\\&\left.\times J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\right\}\end{aligned}\qquad(35)$$

$$\begin{aligned}\sigma_{yy}^{x}(x,y,z)&=\frac{\nu}{1-\nu}\sigma_{zz}^{x}(x,y,z)\\&+\frac{1}{16\pi(1-\nu)^2h^2}\frac{x}{\sqrt{(x^2+y^2)}}\\&\times\left\{(\nu-1)\frac{x^2-3y^2}{x^2+y^2}\right.\\&\times\int_{\lambda=0}^{\lambda=\infty}\lambda[f_{x1}^{x}(\lambda)-f_{x2}^{x}(\lambda)]\\&\times J_3\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\\&-\int_{\lambda=0}^{\lambda=\infty}\lambda[(1+3\nu)f_{x1}^{x}(\lambda)\\&-(1-\nu)f_{x2}^{x}(\lambda)+4\nu f_{x3}^{x}(\lambda)]\\&\left.\times J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\right\}\end{aligned}\qquad(36)$$

$$\begin{aligned}\sigma_{xy}^{x}(x,y,z)&=-\frac{1}{16\pi(1-\nu)h^2}\frac{y}{\sqrt{(x^2+y^2)}}\left\{\frac{y^2-3x^2}{x^2+y^2}\right.\\&\times\int_{\lambda=0}^{\lambda=\infty}\lambda[f_{x1}^{x}(\lambda)-f_{x2}^{x}(\lambda)]\\&\times J_3\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\\&+\int_{\lambda=0}^{\lambda=\infty}\lambda[f_{x1}^{x}(\lambda)+f_{x2}^{x}(\lambda)+2f_{x3}^{x}(\lambda)]\\&\left.\times J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\right\}\end{aligned}\qquad(37)$$

$$\begin{aligned}\sigma_{xx}^{y}(x,y,z)&=\frac{\nu}{1-\nu}\sigma_{zz}^{y}(x,y,z)\\&+\frac{1}{16\pi(1-\nu)^2h^2}\frac{y}{\sqrt{(x^2+y^2)}}\\&\times\left\{(\nu-1)\frac{y^2-3x^2}{x^2+y^2}\right.\\&\times\int_{\lambda=0}^{\lambda=\infty}\lambda[f_{y1}^{y}(\lambda)-f_{y2}^{y}(\lambda)]\\&\times J_3\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\\&-\int_{\lambda=0}^{\lambda=\infty}\lambda[(1+3\nu)f_{y1}^{y}(\lambda)-(1-\nu)\\&\times f_{y2}^{y}(\lambda)+4\nu f_{y3}^{y}(\lambda)]\\&\left.\times J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\right\}\end{aligned}\qquad(38)$$

$$\begin{aligned}\sigma_{yy}^{y}(x,y,z)&=\frac{\nu}{1-\nu}\sigma_{zz}^{y}(x,y,z)\\&+\frac{1}{16\pi(1-\nu)^2h^2}\frac{y}{\sqrt{(x^2+y^2)}}\\&\times\left\{(1-\nu)\frac{y^2-3x^2}{x^2+y^2}\right.\\&\times\int_{\lambda=0}^{\lambda=\infty}\lambda[f_{y1}^{y}(\lambda)-f_{y2}^{y}(\lambda)]\\&\times J_3\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\\&-\int_{\lambda=0}^{\lambda=\infty}\lambda[(3+\nu)f_{y1}^{y}(\lambda)+(1-\nu)\\&\times f_{y2}^{y}(\lambda)+4f_{y3}^{y}(\lambda)]\\&\left.\times J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\right\}\end{aligned}\qquad(39)$$

$$\begin{aligned}\sigma_{xy}^{y}(x,y,z)&=-\frac{1}{16\pi(1-\nu)h^2}\frac{x}{\sqrt{(x^2+y^2)}}\left\{\frac{x^2-3y^2}{x^2+y^2}\right.\\&\times\int_{\lambda=0}^{\lambda=\infty}\lambda[f_{y1}^{y}(\lambda)-f_{y2}^{y}(\lambda)]\\&\times J_3\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\end{aligned}$$

$$+\int_{\lambda=0}^{\lambda=\infty} \lambda[f_{y1}^{y}(\lambda)+f_{y2}^{y}(\lambda)+2f_{y3}^{y}(\lambda)]$$

$$\times J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\Bigg\} \qquad (40)$$

$$u_z^z(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}$$

$$\times\int_{\lambda=0}^{\lambda=\infty}\Bigg\{f_z^z(\lambda)J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)$$

$$-\frac{24(1-\nu)^2}{\lambda^3}$$

$$-\Bigg[12\nu(1-\nu)(\chi+\psi-\chi^2-\psi^2)$$

$$+\tfrac{24}{5}(1-\nu)^2-6(1-\nu)^2$$

$$\times\frac{x^2+y^2}{h^2}\Bigg]\frac{\mathrm{e}^{-\lambda}}{\lambda}\Bigg\}\mathrm{d}\lambda \qquad (41)$$

$$u_x^z(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{x}{\sqrt{(x^2+y^2)}}$$

$$\times\int_{\lambda=0}^{\lambda=\infty}\Bigg\{f_x^z(\lambda)J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)$$

$$-6(1-\nu)^2\frac{\sqrt{(x^2+y^2)}}{h}$$

$$\times(2\chi-1)\frac{\mathrm{e}^{-\lambda}}{\lambda}\Bigg\}\mathrm{d}\lambda \qquad (42)$$

$$u_y^z(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{y}{\sqrt{(x^2+y^2)}}$$

$$\times\int_{\lambda=0}^{\lambda=\infty}\Bigg\{f_y^z(\lambda)J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)$$

$$-6(1-\nu)^2\frac{\sqrt{(x^2+y^2)}}{h}$$

$$\times(2\chi-1)\frac{\mathrm{e}^{-\lambda}}{\lambda}\Bigg\}\mathrm{d}\lambda \qquad (43)$$

$$u_z^x(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{-x}{\sqrt{(x^2+y^2)}}$$

$$\times\int_{\lambda=0}^{\lambda=\infty}\Bigg\{f_z^x(\lambda)J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)$$

$$-6(1-\nu)^2(2\psi-1)\frac{\mathrm{e}^{-\lambda}}{\lambda}\Bigg\}\mathrm{d}\lambda \qquad (44)$$

$$u_x^x(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\Bigg\{\int_{\lambda=0}^{\lambda=\infty}$$

$$\times\Bigg\{[f_{x3}^x(\lambda)+\tfrac{1}{2}(f_{x1}^x(\lambda)+f_{x2}^x(\lambda))]$$

$$\times J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)$$

$$-(\nu-1)[6(1-\nu)(\chi+\psi-2\chi\psi)$$

$$+2(2\nu-3)]\frac{\mathrm{e}^{-\lambda}}{\lambda}\Bigg\}\mathrm{d}\lambda$$

$$+\frac{1}{2}\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}(f_{x1}^x(\lambda)-f_{x2}^x(\lambda))$$

$$\times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\Bigg\} \qquad (45)$$

$$u_y^x(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{-xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}f_y^x(\lambda)$$

$$\times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda \qquad (46)$$

$$u_z^y(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{-y}{\sqrt{(x^2+y^2)}}$$

$$\times\int_{\lambda=0}^{\lambda=\infty}\Bigg\{f_z^y(\lambda)J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)$$

$$-6(1-\nu)^2(2\psi-1)\frac{\mathrm{e}^{-\lambda}}{\lambda}\Bigg\}\mathrm{d}\lambda \qquad (47)$$

$$u_x^y(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{-xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}f_y^x(\lambda)$$

$$\times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda \qquad (48)$$

$$u_y^y(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}$$

$$\times\Bigg\{\int_{\lambda=0}^{\lambda=\infty}\Bigg\{[f_{y3}^y(\lambda)+\tfrac{1}{2}(f_{y1}^y(\lambda)$$

$$+f_{y2}^y(\lambda))]J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)$$

$$-(\nu-1)[6(1-\nu)(\chi+\psi-2\chi\psi)$$

$$+2(2\nu-3)]\frac{\mathrm{e}^{-\lambda}}{\lambda}\Bigg\}\mathrm{d}\lambda$$

$$-\frac{1}{2}\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}(f_{y1}^y(\lambda)-f_{y2}^y(\lambda))$$

$$\times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\Bigg\}, \qquad (49)$$

where the superscript (x, y, z) indicates the direction of the unit load;

$$\chi=\frac{z}{h},\quad \psi=\frac{H}{h};\quad f{:}(\lambda),$$

defined in (23)–(49), are functions of λ, χ and ψ and are defined in [11, 12] and Appendix B.

(a) *Basic features of the solution*

In this section we will discuss the characteristic features of the solution presented above, see expressions (23)–(49). In particular the following properties are demonstrated.

(a) The expressions for the displacement field, eqns (41)–(49), satisfy the displacement equations of equilibrium:

$$\frac{E}{2(1+\nu)(1-2\nu)}u_{j,ij}(x)+\frac{E}{2(1+\nu)}u_{i,jj}(x)$$

$$+F_i(\xi)=0 \quad \begin{cases}\xi=H\mathbf{e}_z,\\ \forall x\neq\xi,\\ i,j=x,y,z.\end{cases} \tag{50}$$

This can be verified by direct differentiation of the convergent integral expressions for the displacements and substitution into (50).

(b) The proposed stress field satisfies the boundary conditions prescribed on the plate surfaces

$$\sigma_{zz}=\sigma_{zx}=\sigma_{zy}=0$$

for

$$\frac{z}{h}=\chi=0 \quad \text{and} \quad \frac{z}{h}=\chi=1.$$

This can be easily seen by inspection of eqns (23)–(31).

(c) The integral of the tractions over the boundary ∂C of a cylinder of arbitrary radius $\rho, \rho>0$ is equal to minus the point load applied at $\xi=H\mathbf{e}_z$.

Letting

$$C=\{(x,y,z)|x^2+y^2\leqslant\rho^2,\ \ 0\leqslant z\leqslant h\}$$

be the cylinder and

$$B_\rho=\{(x,y,z)|x^2+y^2=\rho^2,\ \ 0<z<h\}$$

be its cylindrical surface, the above statement is equivalent to

$$\int_{\partial C}\sigma\cdot\mathbf{n}\,\mathrm{d}A=\int_{B\rho}\sigma\cdot\mathbf{n}\,\mathrm{d}A=-\mathbf{F}, \tag{51}$$

since $\sigma\cdot\mathbf{n}=0$ on the surfaces $\chi=0$, $\chi=1$ of the cylinder, and $\mathbf{F}$ is the applied point load.

The proof was outlined in an earlier paper [2].

(d) The stress and displacement fields have the property:

$$\left.\begin{aligned}\sigma(\mathbf{x})&=O(r^{-2})\\ \mathbf{u}(\mathbf{x})&=O(r^{-1})\end{aligned}\right\}\quad r\to 0$$

$$r=(x^2+y^2+(z-H)^2)^{1/2}>0. \tag{52}$$

In particular, the displacements and stresses of the present solution reduce to the equivalent ones predicted by Kelvin's solution, as the point of application of the load is approached. The proof of the above for the specific case of a displacement component is given in Appendix C. Moreover, in [2] this proof is outlined for a stress component.

The complete proof for all displacement and stress components for concentrated loads along any direction is entirely analogous.

(b) *Far-field analysis of stress and displacement components*

The solution defined by expressions (23)–(49) is formally satisfactory. However, for some of the expressions, a further integration would enable them to lend themselves to an easier physical interpretation.

In particular, the solution in the present form throws no light on the question of the behaviour at points whose distance from the applied load is large in comparison with the plate thickness. By using the integrals defined in Appendix D, the expressions, from which singular terms have been subtracted, (32), (33), (41)–(45), (47), (49), are shown to be composed of two parts of very different character. The first part is a function the value of which decreases as the distance from the source increases, while the second part is a function of a very simple form. Thus, the solution is separated into a local, transitory, or decaying part, which fades away from the neighbourhood of the applied load, and a permanent, or persistent part, which is important in the whole domain occupied by the layer.

$$\begin{aligned}\sigma^z_{xx}(x,y,z)&=\frac{\nu}{1-\nu}\sigma^z_{zz}(x,y,z)\\&+\frac{1}{8\pi(1-\nu)^2h^2}\left\{(1+\nu)\int_{\lambda=0}^{\lambda=\infty}\right.\\&\times\left[\lambda f^z_x(\lambda)-12(1-\nu)^2(2\chi-1)\frac{1}{\lambda}\right]\\&\times J_0\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda+(1-\nu)\frac{y^2-x^2}{R^2}\\&\times\left.\int_{\lambda=0}^{\lambda=\infty}\lambda f^z_x(\lambda)J_2\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda\right\}\\&-\frac{3(1+\nu)}{2\pi h^2}(2\chi-1)\ln\frac{R}{2h}\end{aligned} \tag{53}$$

$$\begin{aligned}\sigma^z_{yy}(x,y,z)&=\frac{\nu}{1-\nu}\sigma^z_{zz}(x,y,z)\\&+\frac{1}{8\pi(1-\nu)^2h^2}\left\{(1+\nu)\int_{\lambda=0}^{\lambda=\infty}\right.\\&\times\left[\lambda f^z_y(\lambda)-12(1-\nu)^2(2\chi-1)\frac{1}{\lambda}\right]\end{aligned}$$

$$\times J_0\left(\frac{\lambda R}{h}\right)\mathrm{d}\lambda + (1-\nu)\frac{x^2-y^2}{R^2}$$

$$\times \int_{\lambda=0}^{\lambda=\infty} \lambda f_y^z(\lambda) J_2\left(\lambda \frac{R}{h}\right)\mathrm{d}\lambda\Bigg\}$$

$$-\frac{3(1+\nu)}{2\pi h^2}(2\chi-1)\ln\frac{\mathrm{R}}{2\mathrm{h}} \tag{54}$$

$$u_z^z(x,y,z) = \frac{1+\nu}{4\pi(1-\nu)hE}$$

$$\times \int_{\lambda=0}^{\lambda=\infty}\Bigg\{f_z^z(\lambda) - \frac{24(1-\nu)^2}{\lambda^3}$$

$$-[12\nu(1-\nu)(\chi+\psi-\chi^2-\psi^2)$$

$$+\tfrac{24}{5}(1-\nu)^2]\frac{1}{\lambda}\Bigg\}J_0\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda$$

$$+\frac{1+\nu}{4\pi(1-\nu)hE}\Bigg\{\ln\frac{R}{2h}\Bigg[6(1-\nu)^2\frac{R^2}{h^2}$$

$$-12\nu(1-\nu)(\chi+\psi-\chi^2-\psi^2)$$

$$-\tfrac{24}{5}(1-\nu)^2\Bigg]-6(1-\nu)^2\frac{R^2}{h^2}\Bigg\} \tag{55}$$

$$u_x^z(x,y,z) = \frac{1+\nu}{4\pi(1-\nu)hE}\frac{x}{R}\int_{\lambda=0}^{\lambda=\infty}$$

$$\times\Bigg\{f_x^z(\lambda) - 12(1-\nu)^2(2\chi-1)\frac{1}{\lambda^2}\Bigg\}$$

$$\times J_1\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda + \frac{3(1-\nu^2)}{2\pi hE}(2\chi-1)\frac{x}{R}$$

$$\times\left[\frac{R}{2h}-\frac{R}{h}\ln\frac{R}{2h}\right] \tag{56}$$

$$u_y^z(x,y,z) = \frac{1+\nu}{4\pi(1-\nu)hE}\frac{y}{R}\int_{\lambda=0}^{\lambda=\infty}$$

$$\times\Bigg\{f_y^z(\lambda) - 12(1-\nu)^2(2\chi-1)\frac{1}{\lambda^2}\Bigg\}$$

$$\times J_1\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda + \frac{3(1-\nu^2)}{2\pi hE}(2\chi-1)\frac{y}{R}$$

$$\times\left[\frac{R}{2h}-\frac{R}{h}\ln\frac{R}{2h}\right] \tag{57}$$

$$u_z^x(x,y,z) = \frac{1+\nu}{4\pi(1-\nu)hE}\frac{-x}{R}\int_{\lambda=0}^{\lambda=\infty}$$

$$\times\Bigg\{f_z^x(\lambda) - 12(1-\nu)^2(2\psi-1)\frac{1}{\lambda^2}\Bigg\}$$

$$\times J_1\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda - \frac{3(1-\nu^2)}{2\pi hE}(2\psi-1)\frac{x}{R}$$

$$\times\left[\frac{R}{2h}-\frac{R}{h}\ln\frac{R}{2h}\right] \tag{58}$$

$$u_x^x(x,y,z) = \frac{1+\nu}{4\pi(1-\nu)hE}\Bigg\{\int_{\lambda=0}^{\lambda=\infty}$$

$$\times\Bigg\{[f_{x_3}^x(\lambda) + \tfrac{1}{2}(f_{x_1}^x(\lambda) + f_{x_2}^x(\lambda))]$$

$$-(\nu-1)[6(1-\nu)(\chi+\psi-2\chi\psi)$$

$$+2(2\nu-3)]\frac{1}{\lambda}\Bigg\}J_0\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda$$

$$+\frac{1}{2}\frac{y^2-x^2}{R}\int_{\lambda=0}^{\lambda=\infty}(f_{x_1}^x(\lambda) - f_{x_2}^x(\lambda))$$

$$\times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\Bigg\}$$

$$+\frac{1+\nu}{4\pi hE}[6(1-\nu)(\chi+\psi-2\chi\psi)$$

$$+2(2\nu-3)]\ln\frac{R}{2h} \tag{59}$$

$$u_z^y(x,y,z) = \frac{1+\nu}{4\pi(1-\nu)hE}\frac{-y}{R}\int_{\lambda=0}^{\lambda=\infty}$$

$$\times\Bigg\{f_z^y(\lambda) - 12(1-\nu)^2(2\psi-1)\frac{1}{\lambda^2}\Bigg\}$$

$$\times J_1\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda - \frac{3(1-\nu^2)}{2\pi hE}(2\psi-1)\frac{y}{R}$$

$$\times\left[\frac{R}{2h}-\frac{R}{h}\ln\frac{R}{2h}\right] \tag{60}$$

$$u_y^y(x,y,z) = \frac{1+\nu}{4\pi(1-\nu)hE}\Bigg\{\int_{\lambda=0}^{\lambda=\infty}$$

$$\times\Bigg\{[f_{y_3}^y(\lambda) + \tfrac{1}{2}(f_{y_1}^y(\lambda) + f_{y_2}^y(\lambda))]$$

$$-(\nu-1)[6(1-\nu)(\chi+\psi-2\chi\psi)$$

$$+2(2\nu-3)]\frac{1}{\lambda}\Bigg\}J_0\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda$$

$$-\frac{1}{2}\frac{y^2-x^2}{R}\int_{\lambda=0}^{\lambda=\infty}(f_{y_1}^y(\lambda) - f_{y_2}^y(\lambda))$$

$$\times J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda\Bigg\}$$

$$+\frac{1+\nu}{4\pi hE}[6(1-\nu)(\chi+\psi-2\chi\psi)$$

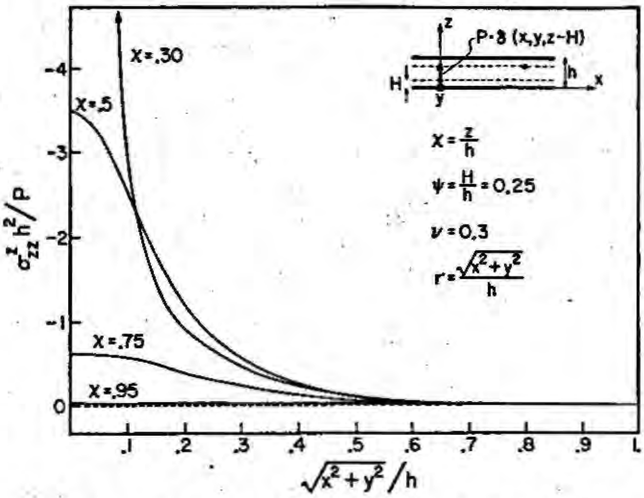

Fig. 2. Variation of the normalized stress $\sigma^z_{zz}h^2/P$ vs the normalized in-plane distance $r' = \sqrt{(x^2+y^2)}/h$.

$$+ 2(2\nu - 3)] \ln \frac{R}{2h}, \tag{61}$$

where $R = \sqrt{(x^2+y^2)}$.

In all of the previous components, the integrands of the transitory part exist and are well behaved for every $\lambda \in [0, \infty)$. And, in particular, the limit of these integrands is null as $\lambda \to 0$ and $\lambda \to \infty$. The remaining components, which have not been considered in this subsection, do not contain persistent elements and the expressions given by (23)–(31), (34)–(40), (46), (48) correspond to the transitory part.

(c) *Numerical evaluation of the solution*

Examples demonstrating some of the features of the three-dimensional solution are presented in Figs 2–4. A point load along the z-direction was applied at a distance $0.25\,h$ from the lower surface of the layer. The variation of the σ^z_{zz} component of the stresses with respect to the normalised in-plane distance $r' = \sqrt{(x^2+y^2)}/h$ measured from the point of application of the load is shown for the cases of $z = 0.95\,h$, $z = 0.75\,h$, $z = 0.5\,h$ and $z = 0.3\,h$. As expected, as $r \to 0$, $(z \to 0.25\,h,\ r' \to 0)$ the stresses reproduce the singular behaviour of the Kelvin state.

Figures 3 and 4 show the variation of the same stress component along the thickness of the plate for different values of the normalized in-plane distance $\sqrt{(x^2+y^2)}/h$ measured from the applied load. At distances close to the load (see Fig. 4, $\sqrt{(x^2+y^2)}/h = 0.05$), the stress changes rapidly from tensile to compressive as the plane of application of the load $(z = 0.25\,h)$ is traversed. As the distance from the load is increased, the tensile portion of the thickness variation diminishes and eventually disappears. It is also worth noting that for distances greater than $0.5\,h$ the thickness variation becomes symmetrically shaped despite the fact that the problem is non-symmetric in the thickness direction,

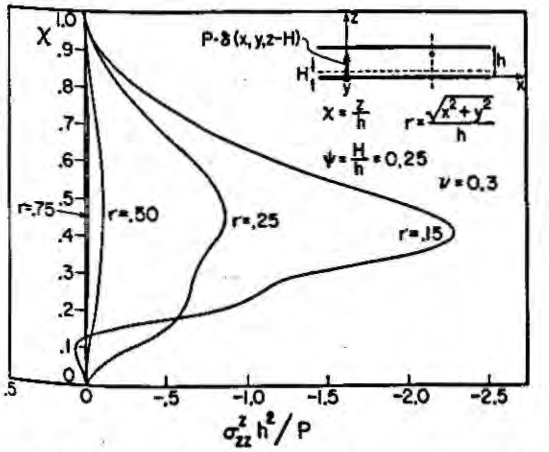

Fig. 3. Variation of the normalized stress $\sigma^z_{zz}h^2/P$ vs the normalized distance $\chi = z/h$ measured from the lower surface of the layer. Different curves correspond to $r' = 0.75$, $r' = 0.50$, $r' = 0.25$, $r' = 0.15$.

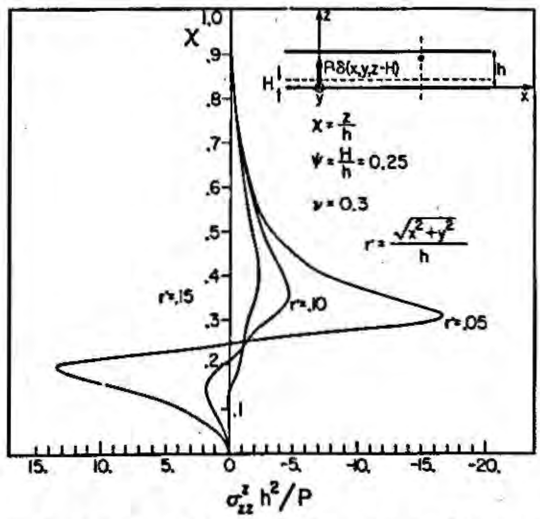

Fig. 4. Variation of the normalized stress $\sigma^z_{zz}h^2/P$ vs the normalized distance $\chi = z/h$ measured from the lower surface of the layer. Different curves correspond to $r' = 0.15$, $r' = 0.10$, $r' = 0.05$.

suggesting that the decay length for the three-dimensional Saint Venant problem is of the order of half the plate thickness.

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APPENDIX A

The stress and displacement functions in the physical space obtained from (20) contains integrals of the following structure:

$$I = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\lambda)\,\mathrm{e}^{-i(\alpha x+\beta y)}\,\mathrm{d}\alpha\,\mathrm{d}\beta, \quad \text{with} \quad \lambda = h\sqrt{(\alpha^2+\beta^2)};$$

depending on $F(\lambda)$, these integrals can be expressed in cylindrical coordinates as

(A.1) Case $F(\lambda) = f(\lambda)$,

$$I = \frac{2\pi}{h^2}\int_{\lambda=0}^{\lambda=\infty} \lambda f(\lambda) J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda.$$

(A.2) Case $F(\lambda) = \alpha f(\lambda)$,

$$I = \frac{1}{h^3}\frac{-2\pi i x}{\sqrt{(x^2+y^2)}}\int_{\lambda=0}^{\lambda=\infty} \lambda^2 f(\lambda) J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda.$$

(A.3) Case $F(\lambda) = \beta f(\lambda)$,

$$I = \frac{1}{h^3}\frac{-2\pi i y}{\sqrt{(x^2+y^2)}}\int_{\lambda=0}^{\lambda=\infty} \lambda^2 f(\lambda) J_1\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda.$$

(A.4) Case $F(\lambda) = \alpha^2 f(\lambda)$,

$$I = \frac{\pi}{h^4}\int_{\lambda=0}^{\lambda=\infty} \lambda^3 f(\lambda) J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda + \frac{\pi}{h^4}\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty} \lambda^3 f(\lambda) J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda,$$

(A.5) Case $F(\lambda) = \beta^2 f(\lambda)$,

$$I = \frac{\pi}{h^4}\int_{\lambda=0}^{\lambda=\infty} \lambda^3 f(\lambda) J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda - \frac{\pi}{h^4}\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty} \lambda^3 f(\lambda) J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda.$$

(A.6) Case $F(\lambda) = \alpha\beta f(\lambda)$,

$$I = -\frac{2\pi}{h^4}\frac{xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty} \lambda^3 f(\lambda) J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)\mathrm{d}\lambda.$$

APPENDIX B

Expressions for $f_{\cdot}^{\cdot}(\lambda)$, defined in (23)–(49), are functions of λ, χ and ψ. It is found [1, 2] that they are given by:

$$f_{zz}^{1}(\lambda) = \frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{\lambda^4\chi(1-\psi)\sinh[\lambda(\chi-\psi)] + \frac{\lambda^3}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)]$$
$$+((4\nu-3)\chi+\psi-1)\cosh[\lambda(\chi-\psi)] - \chi\psi\cosh[\lambda(2-\chi-\psi)]]$$

$$
\begin{aligned}
&+\frac{\lambda^2}{4}[((4\nu-3)\chi+4(1-\nu)-\psi)\sinh[\lambda(\chi+\psi)]+(4(1-\nu)+\chi-\psi)\sinh[\lambda(\chi-\psi)]\\
&+((4\nu-3)\chi-\psi)\sinh[\lambda(2-\chi-\psi)]+(\psi-\chi)\sinh[\lambda(2+\chi-\psi)]]\\
&-(1-\nu)\frac{\lambda}{2}[-\cosh[\lambda(\chi+\psi)]+\cosh[\lambda(\chi-\psi)]+\cosh[\lambda(2-\chi-\psi)]-\cosh[\lambda(2+\chi-\psi)]]\Big\}\\
&+\{\lambda^2(\chi-\psi)\sinh[\lambda(\chi-\psi)]-\lambda 2(1-\nu)\cosh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^z_{zx}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{&\lambda^4\chi(\psi-1)\cosh[\lambda(\chi-\psi)]-\frac{\lambda^3}{2}[(1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)]\\
&+((4\nu-3)\chi-\psi+1)\sinh[\lambda(\chi-\psi)]+\chi\psi\sinh[\lambda(2-\chi-\psi)]]\\
&-\frac{\lambda^2}{4}[((4\nu-3)\chi+2(1-2\nu)+\psi)\cosh[\lambda(\chi+\psi)]+(2(2\nu-1)+\chi-\psi)\cosh[\lambda(\chi-\psi)]\\
&+((3-4\nu)\chi-\psi)\cosh[\lambda(2-\chi-\psi)]+(\psi-\chi)\cosh[\lambda(2+\chi-\psi)]]\\
&-(2\nu-1)\frac{\lambda}{4}[\sinh[\lambda(\chi+\psi)]+\sinh[\lambda(\chi-\psi)]+\sinh[\lambda(2-\chi-\psi)]-\sinh[\lambda(2+\chi-\psi)]]\Big\}\\
&+\{-\lambda^2(\chi-\psi)\cosh[\lambda(\chi-\psi)]-\lambda(2\nu-1)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^z_{zy}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{&\lambda^4\chi(\psi-1)\cosh[\lambda(\chi-\psi)]-\frac{\lambda^3}{2}[(1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)]\\
&+((4\nu-3)\chi-\psi+1)\sinh[\lambda(\chi-\psi)]+\chi\psi\sinh[\lambda(2-\chi-\psi)]]\\
&-\frac{\lambda^2}{4}[((4\nu-3)\chi+2(1-2\nu)+\psi)\cosh[\lambda(\chi+\psi)]+(2(2\nu-1)+\chi-\psi)\cosh[\lambda(\chi-\psi)]\\
&+((3-4\nu)\chi-\psi)\cosh[\lambda(2-\chi-\psi)]+(\psi-\chi)\cosh[\lambda(2+\chi-\psi)]]\\
&-(2\nu-1)\frac{\lambda}{4}[\sinh[\lambda(\chi+\psi)]+\sinh[\lambda(\chi-\psi)]+\sinh[\lambda(2-\chi-\psi)]-\sinh[\lambda(2+\chi-\psi)]]\Big\}\\
&+\{-\lambda^2(\chi-\psi)\cosh[\lambda(\chi-\psi)]-\lambda(2\nu-1)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^x_{zz}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{&\lambda^4\chi(1-\psi)\cosh[\lambda(\chi-\psi)]-\frac{\lambda^3}{2}[(1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)]\\
&+((4\nu-3)\chi-\psi+1)\sinh[\lambda(\chi-\psi)]+\chi\psi\sinh[\lambda(2-\chi-\psi)]]\\
&+\frac{\lambda^2}{4}[((4\nu-3)\chi+2(1-2\nu)+\psi)\cosh[\lambda(\chi+\psi)]+(2(2\nu-1)+\chi-\psi)\cosh[\lambda(\chi-\psi)]\\
&+((3-4\nu)\chi-\psi)\cosh[\lambda(2-\chi-\psi)]+(\psi-\chi)\cosh[\lambda(2+\chi-\psi)]]\\
&-(2\nu-1)\frac{\lambda}{4}[\sinh[\lambda(\chi+\psi)]+\sinh[\lambda(\chi-\psi)]+\sinh[\lambda(2-\chi-\psi)]-\sinh[\lambda(2+\chi-\psi)]]\Big\}\\
&+\{\lambda^2(\chi-\psi)\cosh[\lambda(\chi-\psi)]-\lambda(2\nu-1)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^x_{zx_1}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{&(1-\nu)\frac{\lambda}{2}[\cosh[\lambda(\chi+\psi)]-\cosh[\lambda(\chi-\psi)]-\cosh[\lambda(2-\chi-\psi)]\\
&+\cosh[\lambda(2+\chi-\psi)]]\Big\}+\{-\lambda 2(1-\nu)\cosh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^x_{zx_2}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{&\lambda^4\chi(\psi-1)\sinh[\lambda(\chi-\psi)]+\frac{\lambda^3}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)]\\
&+((4\nu-3)\chi+\psi-1)\cosh[\lambda(\chi-\psi)]-\chi\psi\cosh[\lambda(2-\chi-\psi)]]\\
&+\frac{\lambda^2}{4}[((3-4\nu)\chi+4(\nu-1)+\psi)\sinh[\lambda(\chi+\psi)]+(4(\nu-1)-\chi+\psi)\sinh[\lambda(\chi-\psi)]\\
&+((3-4\nu)\chi+\psi)\sinh[\lambda(2-\chi-\psi)]+(\chi-\psi)\sinh[\lambda(2+\chi-\psi)]]\}\\
&+\{\lambda^2(\psi-\chi)\sinh[\lambda(\chi-\psi)]\Big\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
f^x_{zx_3}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\left\{-2(1-\nu)\lambda^3\frac{\sinh(\lambda\chi)\cosh[\lambda(1-\psi)]}{\sinh(\lambda)}\right\};
$$

$$
\begin{aligned}
f^x_{zy}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\Big\{&\lambda^4\chi(\psi-1)\sinh[\lambda(\chi-\psi)]+\frac{\lambda^3}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)]\\
&+((4\nu-3)\chi+\psi-1)\cosh[\lambda(\chi-\psi)]-\chi\psi\cosh[\lambda(2-\chi-\psi)]]\\
&+\frac{\lambda^2}{4}[((3-4\nu)\chi+4(\nu-1)+\psi)\sinh[\lambda(\chi+\psi)]+(4(\nu-1)-\chi+\psi)\sinh[\lambda(\chi-\psi)]\\
&+((3-4\nu)\chi+\psi)\sinh[\lambda(2-\chi-\psi)]+(\chi-\psi)\sinh[\lambda(2+\chi-\psi)]]
\end{aligned}
$$

$$
+2(1-\nu)\lambda^3\frac{\sinh(\lambda\chi)\cosh[\lambda(1-\psi)]}{\sinh(\lambda)}\Bigg\} + \{\lambda^2(\psi-\chi)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
$$

$$
\begin{aligned}
f^{y}_{zz}(\lambda) = \frac{1}{\sinh^2(\lambda)-\lambda^2}\Bigg\{ & \lambda^4\chi(1-\psi)\cosh[\lambda(\chi-\psi)] - \frac{\lambda^3}{2}[(1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)] \\
& + ((4\nu-3)\chi-\psi+1)\sinh[\lambda(\chi-\psi)] + \chi\psi\sinh[\lambda(2-\chi-\psi)]] \\
& + \frac{\lambda^2}{4}[((4\nu-3)\chi+2(1-2\nu)+\psi)+\cosh[\lambda(\chi+\psi)] + (2(2\nu-1)+\chi-\psi)\cosh[\lambda(\chi-\psi)] \\
& + ((3-4\nu)\chi-\psi)\cosh[\lambda(2-\chi-\psi)] + (\psi-\chi)\cosh[\lambda(2+\chi-\psi)]] \\
& - (2\nu-1)\frac{\lambda}{4}[\sinh[\lambda(\chi+\psi)] + \sinh[\lambda(\chi-\psi)] + \sinh[\lambda(2-\chi-\psi)] - \sinh[\lambda(2+\chi-\psi)]]\Bigg\} \\
& + \{\lambda^2(\chi-\psi)\cosh[\lambda(\chi-\psi)] - \lambda(2\nu-1)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^{y}_{zy_1}(\lambda) = \frac{1}{\sinh^2(\lambda)-\lambda^2}\Bigg\{ & (1-\nu)\frac{\lambda}{2}[\cosh[\lambda(\chi+\psi)] - \cosh[\lambda(\chi-\psi)] - \cosh[\lambda(2-\chi-\psi)] + \cosh[\lambda(2+\chi-\psi)]]\Bigg\} \\
& + \{-\lambda 2(1-\nu)\cosh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^{y}_{xy_2}(\lambda) = \frac{1}{\sinh^2(\lambda)-\lambda^2}\Bigg\{ & \lambda^4\chi(\psi-1)\sinh[\lambda(\chi-\psi)] + \frac{\lambda^3}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)] \\
& + ((4\nu-3)\chi+\psi-1)\cosh[\lambda(\chi-\psi)] - \chi\psi\cosh[\lambda(2-\chi-\psi)]] \\
& + \frac{\lambda^2}{4}[((3-4\nu)\chi+4(\nu-1)+\psi)+\sinh[\lambda(\chi+\psi)] + (4(\nu-1)-\chi+\psi)\sinh[\lambda(\chi-\psi)] \\
& + ((3-4\nu)\chi+\psi)\sinh[\lambda(2-\chi-\psi)] + (\chi-\psi)\sinh[\lambda(2+\chi-\psi)]]\Bigg\} \\
& + \{\lambda^2(\psi-\chi)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
f^{y}_{zy_3}(\lambda) = \frac{1}{\sinh^2(\lambda)-\lambda^2}\left\{-2(1-\nu)\lambda^3\frac{\sinh(\lambda\chi)\cosh[\lambda(1-\psi)]}{\sinh(\lambda)}\right\};
$$

$$
\begin{aligned}
f^{z}_{y}(\lambda) = \frac{1}{\sinh^2(\lambda)-\lambda^2}\Bigg\{ & \lambda^3\chi(\psi-1)\sinh[\lambda(\chi-\psi)] - \frac{\lambda^2}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)] \\
& + (4\nu-3)(\chi+\psi-1)\cosh[\lambda(\chi-\psi)] - \chi\psi\cosh[\lambda(2-\chi-\psi)]] \\
& + \frac{\lambda}{4}[(4\nu-3)(\psi-\chi)\sinh[\lambda(\chi+\psi)] + (8(2\nu-1)(\nu-1)-\chi+\psi)\sinh[\lambda(\chi-\psi)] \\
& + (4\nu-3)(\psi-\chi)\sinh[\lambda(2-\chi-\psi)] + (\chi-\psi)\sinh[\lambda(2+\chi-\psi)]] + (2\nu-1)(\nu-1)[\cosh[\lambda(\chi+\psi)] \\
& - \cosh[\lambda(2-\chi-\psi)]]\Bigg\} + \{\lambda(\psi-\chi)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^{z}_{x}(\lambda) = \frac{1}{\sinh^2(\lambda)-\lambda^2}\Bigg\{ & \lambda^3\chi(\psi-1)\sinh[\lambda(\chi-\psi)] - \frac{\lambda^2}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)] \\
& + (4\nu-3)(\chi+\psi-1)\cosh[\lambda(\chi-\psi)] - \chi\psi\cosh[\lambda(2-\chi-\psi)]] \\
& + \frac{\lambda}{4}[(4\nu-3)(\psi-\chi)\sinh[\lambda(\chi+\psi)] + (8(2\nu-1)(\nu-1)-\chi+\psi)\sinh[\lambda(\chi-\psi)] \\
& + (4\nu-3)(\psi-\chi)\sinh[\lambda(2-\chi-\psi)] + (\chi-\psi)\sinh[\lambda(2+\chi-\psi)]] \\
& + (2\nu-1)(\nu-1)[\cosh[\lambda(\chi+\psi)] - \cosh[\lambda(2-\chi-\psi)]]\Bigg\} + \{\lambda(\psi-\chi)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^{z}_{z}(\lambda) = \frac{1}{\sinh^2(\lambda)-\lambda^2}\Bigg\{ & \lambda^3\chi(1-\psi)\cosh[\lambda(\chi-\psi)] + \frac{\lambda^2}{2}[(4\nu-3)(1+\chi-\psi)\sinh[\lambda(\chi-\psi)] \\
& + (1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)] + \chi\psi\sinh[\lambda(2-\chi-\psi)]] \\
& + \frac{\lambda}{4}[(2(8\nu^2-12\nu+5)+\chi-\psi)\cosh[\lambda(\chi-\psi)] + (3-4\nu)(2-\chi-\psi)\cosh[\lambda(\chi+\psi)] \\
& + (3-4\nu)(\chi+\psi)\cosh[\lambda(2-\chi-\psi)] + (\psi-\chi)\cosh[\lambda(2+\chi-\psi)]] \\
& + \tfrac{1}{4}[(8\nu^2-12\nu+5)\sinh[\lambda(\chi+\psi)] + (8\nu^2-12\nu+5)\sinh[\lambda(2-\chi-\psi)] \\
& + (4\nu-3)\sinh[\lambda(\chi-\psi)] - (4\nu-3)\sinh[\lambda(2+\chi-\psi)]]\Bigg\} \\
& + \{\lambda(\chi-\psi)\cosh[\lambda(\chi-\psi)] + (4\nu-3)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
f^{x}_{y}(\lambda) = \frac{1}{\sinh^2(\lambda)-\lambda^2}\Bigg\{\lambda^3\chi(\psi-1)\cosh[\lambda(\chi-\psi)] + \frac{\lambda^2}{2}[((2\nu-1)+\chi\psi-\chi-\psi)\sinh[\lambda(\chi+\psi)]
$$

$$
\begin{aligned}
&+((2\nu-1)+(3-4\nu)(\psi-\chi))\sinh[\lambda(\chi-\psi)]+(2(\nu-1)+\chi\psi)\sinh[\lambda(2-\chi-\psi)]\\
&+2(\nu-1)\sinh[\lambda(2+\chi-\psi)]]-\frac{\lambda}{4}[(2(8\nu^2-12\nu+5)+\chi-\psi)\cosh[\lambda(\chi-\psi)]\\
&+(3-4\nu)(2-\chi-\psi)\cosh[\lambda(\chi+\psi)]+(3-4\nu)(\chi+\psi)\cosh[\lambda(2-\chi-\psi)]+(\psi-\chi)\cosh[\lambda(2+\chi-\psi)]]\\
&+\tfrac{1}{4}[(8\nu^2-8\nu+1)\sinh[\lambda(\chi+\psi)]+(8\nu^2-8\nu+1)\sinh[\lambda(2-\chi-\psi)]+\sinh[\lambda(\chi-\psi)]-\sinh[\lambda(2+\chi-\psi)]]\\
&\left.+4(1-\nu)\lambda^2\frac{\cosh^2(\lambda)\cosh(\lambda\chi)}{\sinh(\lambda)}\cosh[\lambda(1-\psi)]\right\}+\{-\lambda(\chi-\psi)\cosh[\lambda(\chi-\psi)]+\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^x_{x1}(\lambda)=&\frac{1}{\sinh^2(\lambda)-\lambda^2}\left\{\lambda^3\chi(\psi-1)\cosh[\lambda(\chi-\psi)]+\frac{\lambda^2}{2}[-(3-4\nu)(1+\chi-\psi)\sinh[\lambda(\chi-\psi)]\right.\\
&+(1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)]+\chi\psi\sinh[\lambda(2-\chi-\psi)]]\\
&-\frac{\lambda}{4}[(2(8\nu^2-12\nu+5)+\chi-\psi)\cosh[\lambda(\chi-\psi)]+(3-4\nu)(2-\chi-\psi)\cosh[\lambda(\chi+\psi)]\\
&+(3-4\nu)(\chi+\psi)\cosh[\lambda(2-\chi-\psi)]+(\psi-\chi)\cosh[\lambda(2+\chi-\psi)]]\\
&+\tfrac{1}{4}[(8\nu^2-8\nu+1)\sinh[\lambda(\chi+\psi)]+(8\nu^2-8\nu+1)\sinh[\lambda(2-\chi-\psi)]\\
&\left.+\sinh[\lambda(\chi-\psi)]-\sinh[\lambda(2+\chi-\psi)]]\right\}+\{-\lambda(\chi-\psi)\cosh[\lambda(\chi-\psi)]+\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
f^x_{x2}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\left[-4(1-\nu)\lambda^2\frac{\cosh(\lambda\chi)\cosh[\lambda(1-\psi)]}{\sinh(\lambda)}\right];
$$

$$
\begin{aligned}
f^x_{x3}(\lambda)=&\frac{1}{\sinh^2(\lambda)-\lambda^2}\{(1-\nu)[\sinh[\lambda(\chi+\psi)]-\sinh[\lambda(\chi-\psi)]+\sinh[\lambda(2-\chi-\psi)]+\sinh[\lambda(2+\chi-\psi)]]\}\\
&+\{-4(1-\nu)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^x_z(\lambda)=&\frac{1}{\sinh^2(\lambda)-\lambda^2}\left\{\lambda^3\chi(1-\psi)\sinh[\lambda(\chi-\psi)]-\frac{\lambda^2}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)]\right.\\
&+(3-4\nu)(1-\chi-\psi)\cosh[\lambda(\chi-\psi)]-\chi\psi\cosh[\lambda(2-\chi-\psi)]]-\frac{\lambda}{4}[(3-4\nu)(\chi-\psi)\sinh[\lambda(\chi+\psi)]\\
&+(8(2\nu-1)(\nu-1)-\chi+\psi)\sinh[\lambda(\chi-\psi)]+(3-4\nu)(\chi-\psi)\sinh[\lambda(2-\chi-\psi)]\\
&\left.+(\chi-\psi)\sinh[\lambda(2+\chi-\psi)]]+(2\nu-1)(\nu-1)[\cosh[\lambda(\chi+\psi)]-\cosh[\lambda(2-\chi-\psi)]]\right\}\\
&+\{-\lambda(\psi-\chi)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^y_{y1}(\lambda)=&\frac{1}{\sinh^2(\lambda)-\lambda^2}\left\{\lambda^3\chi(\psi-1)\cosh[\lambda(\chi-\psi)]+\frac{\lambda^2}{2}[-(3-4\nu)(1+\chi-\psi)\sinh[\lambda(\chi-\psi)]\right.\\
&+(1-\chi)(1-\psi)\sinh[\lambda(\chi+\psi)]+\chi\psi\sinh[\lambda(2-\chi-\psi)]]\\
&-\frac{\lambda}{4}[(2(8\nu^2-12\nu+5)+\chi-\psi)\cosh[\lambda(\chi-\psi)]+(3-4\nu)(2-\chi-\psi)\cosh[\lambda(\chi+\psi)]\\
&+(3-4\nu)(\chi+\psi)\cosh[\lambda(2-\chi-\psi)]+(\psi-\chi)\cosh[\lambda(2+\chi-\psi)]]\\
&+\tfrac{1}{4}[(8\nu^2-8\nu+1)\sinh[\lambda(\chi+\psi)]+(8\nu^2-8\nu+1)\sinh[\lambda(2-\chi-\psi)]+\sinh[\lambda(\chi-\psi)]\\
&\left.-\sinh[\lambda(2+\chi-\psi)]]\right\}+\{-\lambda(\chi-\psi)\cosh[\lambda(\chi-\psi)]+\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
f^y_{y2}(\lambda)=\frac{1}{\sinh^2(\lambda)-\lambda^2}\left[-4(1-\nu)\lambda^2\frac{\cosh(\lambda\chi)\cosh[\lambda(1-\psi)]}{\sinh(\lambda)}\right];
$$

$$
\begin{aligned}
f^y_{y3}(\lambda)=&\frac{1}{\sinh^2(\lambda)-\lambda^2}\{(1-\nu)[\sinh[\lambda(\chi+\psi)]-\sinh[\lambda(\chi-\psi)]+\sinh[\lambda(2-\chi-\psi)]+\sinh[\lambda(2+\chi-\psi)]]\}\\
&+\{-4(1-\nu)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

$$
\begin{aligned}
f^y_z(\lambda)=&\frac{1}{\sinh^2(\lambda)-\lambda^2}\left\{\lambda^3\chi(1-\psi)\sinh[\lambda(\chi-\psi)]-\frac{\lambda^2}{2}[(1-\chi)(1-\psi)\cosh[\lambda(\chi+\psi)]\right.\\
&+(3-4\nu)(1-\chi-\psi)\cosh[\lambda(\chi-\psi)]-\chi\psi\cosh[\chi(2-\chi-\psi)]]\\
&-\frac{\lambda}{4}[(3-4\nu)(\chi-\psi)\sinh[\lambda(\chi+\psi)]+(8(2\nu-1)(\nu-1)-\chi+\psi)\sinh[\lambda(\chi-\psi)]\\
&+(3-4\nu)(\chi-\psi)\sinh[\lambda(2-\chi-\psi)]+(\chi-\psi)\sinh[\lambda(2+\chi-\psi)]]+(2\nu-1)(\nu-1)[\cosh[\lambda(\chi+\psi)]\\
&\left.-\cosh[\lambda(2-\chi-\psi)]]\right\}+\{-\lambda(\psi-\chi)\sinh[\lambda(\chi-\psi)]\}\mathscr{H}(\chi-\psi);
\end{aligned}
$$

where

$$\chi=\frac{z}{h} \quad \text{and} \quad \psi=\frac{H}{h}.$$

APPENDIX C

In this appendix the proof of the property expressed by (52) is outlined by making use of a specific displacement component corresponding to a unit load in the positive y direction.

We consider the displacement component u_y^y given in (49). As it was stated in the end of Sec. 5, the integrand

$$\left\{[f_{y_3}^y(\lambda)+\tfrac{1}{2}(f_{y_1}^y(\lambda)+f_{y_2}^y(\lambda))]J_0\left(\lambda\frac{R}{h}\right)-A\frac{e^{-\lambda}}{\lambda}\right\} \quad \text{and} \quad (f_{y_1}^y(\lambda)-f_{y_2}^y(\lambda))J_2\left(\lambda\frac{R}{h}\right)$$

of (49) are well behaved for every $\lambda\in[0,\infty)$ and in particular

$$\lim_{\lambda\to 0}\left\{[f_{y_3}^y(\lambda)+\tfrac{1}{2}(f_{y_1}^y(\lambda)+f_{y_2}^y(\lambda))]J_0\left(\lambda\frac{R}{h}\right)-A\frac{e^{-\lambda}}{\lambda}\right\}=0$$

$$\lim_{\lambda\to 0}(f_{y_1}^y(\lambda)-f_{y_2}^y(\lambda))J_2\left(\lambda\frac{R}{h}\right)=0.$$

This allows us to replace the integral of (49) by its Cauchy principal value:

$$\begin{aligned}u_y^y(x,y,z)=\frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}&\left\{\int_{\lambda=0+}^{\lambda=\infty}\left\{[f_{y_3}^y(\lambda)+\tfrac{1}{2}(f_{y_1}^y(\lambda)+f_{y_2}^y(\lambda))]\right.\right.\\&\left.\times J_0\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)-(\nu-1)[6(1-\nu)(\chi+\psi-2\chi\psi)+2(2\nu-3)]\frac{e^{-\lambda}}{\lambda}\right\}d\lambda\\&\left.-\frac{1}{2}\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0+}^{\lambda=\infty}(f_{y_1}^y(\lambda)-f_{y_2}^y(\lambda))J_2\left(\frac{\lambda\sqrt{(x^2+y^2)}}{h}\right)d\lambda\right\},\end{aligned}\tag{C1}$$

where

$$R=(x^2+y^2)^{1/2}>0, \quad \chi=\frac{z}{h}, \quad \psi=\frac{H}{h},$$

and

$$A=(\nu-1)[6(1-\nu)(\chi+\psi-2\chi\psi)+2(2\nu-3)].$$

The expressions for $f_{y_1}^y(\lambda), f_{y_2}^y(\lambda), f_{y_3}^y(\lambda)$ are given in Appendix B.

We now choose to replace the hyperbolic functions in $f_{y_1}^y(\lambda), f_{y_2}^y(\lambda), f_{y_3}^y(\lambda)$ by their equivalent exponential forms. By doing so, these can be represented as:

$$\begin{aligned}f_{y_1}^y(\lambda)=\frac{1}{1+[e^{-4\lambda}-(2+4\lambda^2)e^{-2\lambda}]}&\left\{2\lambda^3\chi(\psi-1)(e^{-\lambda(2-\chi+\psi)}+e^{-\lambda(2+\chi-\psi)})+\lambda^2[(1-\chi)(1-\psi)(e^{-\lambda(2-\chi-\psi)}-e^{-\lambda(2+\chi+\psi)})\right.\\&+(3-4\nu)(\psi-\chi-1)(e^{-\lambda(2-\chi+\psi)}-e^{-\lambda(2+\chi-\psi)})+\chi\psi(e^{-\lambda(\chi+\psi)}-e^{-\lambda(4-\chi-\psi)})]\\&-\frac{\lambda}{2}[(2(8\nu^2-12\nu+5)+\chi-\psi)(e^{-\lambda(2-\chi+\psi)}+e^{-\lambda(2+\chi-\psi)})+(\psi-\chi)(e^{\lambda(\chi-\psi)}+e^{-\lambda(4+\chi-\psi)})\\&+(3-4\nu)(2-\chi-\psi)(e^{-\lambda(2-\chi-\psi)}+e^{-\lambda(2+\chi+\psi)})+(3-4\nu)(\chi+\psi)(e^{-\lambda(\chi+\psi)}+e^{-\lambda(4-\chi-\psi)})]\\&+\tfrac{1}{2}[(8\nu^2-8\nu+1)(e^{-\lambda(2-\chi-\psi)}-e^{-\lambda(2+\chi+\psi)}+e^{-\lambda(\chi+\psi)}-e^{-\lambda(4-\chi-\psi)})+e^{-\lambda(2-\chi+\psi)}\\&-e^{-\lambda(2+\chi-\psi)}-e^{\lambda(\chi-\psi)}+e^{-\lambda(4+\chi-\psi)}]+\left\langle-(\chi-\psi)\frac{\lambda}{2}[e^{\lambda(\chi-\psi)}+e^{-\lambda(\chi-\psi)}+e^{-\lambda(4-\chi+\psi)}+e^{-\lambda(4+\chi-\psi)}\right.\\&-2e^{-\lambda(2-\chi+\psi)}-2e^{-\lambda(2+\chi-\psi)}]+2(\chi-\psi)\lambda^3[e^{-\lambda(2-\chi+\psi)}+e^{-\lambda(2+\chi-\psi)}]\\&+\tfrac{1}{2}[e^{\lambda(\chi-\psi)}-e^{-\lambda(\chi-\psi)}+e^{-\lambda(4-\chi+\psi)}-e^{-\lambda(4+\chi-\psi)}-2e^{-\lambda(2-\chi+\psi)}+2e^{-\lambda(2+\chi-\psi)}]\\&\left.\left.-2\lambda^2[e^{-\lambda(2-\chi+\psi)}-e^{-\lambda(2+\chi-\psi)}]\right\rangle\mathscr{H}(\chi-\psi)\right\}\end{aligned}\tag{C2a}$$

$$f_{y_2}^y(\lambda)=\frac{1}{1+[e^{-4\lambda}-(2+4\lambda^2)e^{-2\lambda}]}\frac{1}{1-e^{-2\lambda}}8(\nu-1)\lambda^2(e^{-\lambda(2-\chi+\psi)}+e^{-\lambda(4-\chi-\psi)}+e^{-\lambda(2+\chi+\psi)}+e^{-\lambda(4+\chi-\psi)})\tag{C2b}$$

$$\begin{aligned}f_{y_3}^y(\lambda)=\frac{1}{1+[e^{-4\lambda}-(2+4\lambda^2)e^{-2\lambda}]}&\{2(1-\nu)[e^{-\lambda(2-\chi-\psi)}-e^{-\lambda(2+\chi+\psi)}-e^{-\lambda(2-\chi+\psi)}+e^{-\lambda(2+\chi-\psi)}+e^{-\lambda(\chi+\psi)}-e^{-\lambda(4-\chi-\psi)}\\&+e^{\lambda(\chi-\psi)}-e^{-\lambda(4+\chi-\psi)}]+\langle-2(1-\nu)[e^{\lambda(\chi-\psi)}-e^{-\lambda(\chi-\psi)}+e^{-\lambda(4-\chi+\psi)}-e^{-\lambda(4+\chi-\psi)}\\&-2e^{-\lambda(2-\chi+\psi)}+2e^{-\lambda(2+\chi-\psi)}]+8(1-\nu)\lambda^2[e^{-\lambda(2-\chi+\psi)}-e^{-\lambda(2+\chi-\psi)}]\rangle\mathscr{H}(\chi-\psi)\}.\end{aligned}\tag{C2c}$$

We first observe that (C2) involves only negative exponentials in λ (the positive ones, which exist for $(\chi-\psi)>0$, cancel out).

Also the factors $(1+[\mathrm{e}^{-4\lambda}-(2+4\lambda^2)\mathrm{e}^{-2\lambda}])^{-1}$ and $(1-\mathrm{e}^{-2\lambda})^{-1}$ can be expanded, in an infinite convergent series, by means of the binomial theorem for every value of $\lambda\in(0,\infty)$. This is true since

$$|\mathrm{e}^{-4\lambda}-(2+4\lambda^2)\mathrm{e}^{-2\lambda}|<1,\quad |\mathrm{e}^{-2\lambda}|<1,\quad \forall\lambda\in(0,\infty),$$

that is, for every λ in the range of integration (C1).

If the expansion is performed, for instance $\chi>\psi$, the resulting terms can be expressed as:

$$f^{y}_{y1}(\lambda,\chi,\psi)=-\frac{\lambda}{2}(\chi-\psi)\mathrm{e}^{-\lambda(\chi-\psi)}-\tfrac{1}{2}\mathrm{e}^{-\lambda(\chi-\psi)}+\sum_{n=0}^{\infty}a_{n1}\lambda^n\mathrm{e}^{-\lambda p_{n1}(\chi-\psi,\psi)}$$

$$f^{y}_{y2}(\lambda,\chi,\psi)=\sum_{n=2}^{\infty}a_{n2}\lambda^n\mathrm{e}^{-\lambda p_{n2}(\chi-\psi,\psi)}$$

$$f^{y}_{y3}(\lambda,\chi,\psi)=2(1-\nu)\mathrm{e}^{-\lambda(\chi-\psi)}+\sum_{n=0}^{\infty}a_{n3}\lambda^n\mathrm{e}^{-\lambda p_{n3}(\chi-\psi,\psi)}.$$

The displacement thus can be expressed as:

$$u^{y}_{y}=\frac{1+\nu}{4\pi(1-\nu)Eh}\left\{-\frac{1}{4}(\chi-\psi)\int_{0+}^{\infty}\chi\mathrm{e}^{-\lambda(\chi-\psi)}J_0\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda+[2(1-\nu)-\tfrac{1}{4}]\int_{0+}^{\infty}\mathrm{e}^{-\lambda(\chi-\psi)}J_0\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda\right.$$
$$+\int_{0+}^{\infty}\left[\sum_{n=0}^{\infty}a_{n4}\lambda^n\mathrm{e}^{-\lambda p_{n4}(\chi-\psi,\psi)}J_0\left(\lambda\frac{R}{h}\right)-A\frac{\mathrm{e}^{-\lambda}}{\lambda}\right]\mathrm{d}\lambda-\frac{1}{2}\frac{y^2-x^2}{x^2+y^2}\left[-\tfrac{1}{2}(\chi-\psi)\int_{0+}^{\infty}\lambda\mathrm{e}^{-\lambda(\chi-\psi)}J_2\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda\right.$$
$$\left.\left.-\frac{1}{2}\int_{0+}^{\infty}\mathrm{e}^{-\lambda(\chi-\psi)}J_2\left(\lambda\frac{R}{h}\right)\mathrm{d}\lambda+\sum_{n=0}^{\infty}a_{n5}\int_{0+}^{\infty}\lambda^n\mathrm{e}^{-\lambda p_{n5}(\chi-\psi,\psi)}J_2\left(\lambda\frac{R}{h}\right)\right]\mathrm{d}\lambda\right]\right\},$$

where $p_{ni}(\chi-\psi,\psi)$, $\forall i=1,2,3,\ldots\ \forall n=0,1,2,\ldots$, are linear functions of $(\chi-\psi)$ and ψ, such that $p_{ni}(\chi-\psi,\psi)>0\ \forall\chi,\psi\in[0,1]$ and in particular $\lim_{\chi\to\psi}p_{ni}(\chi-\psi,\psi)\neq0$. Also a_{ni}, $\forall i=1,2,3,\ldots\ \forall n=0,1,2,\ldots$, are linear functions of χ and ψ.

Letting

$$r=[R^2+(z-H)^2]^{1/2}$$

$$\phi=\cos^{-1}\frac{R}{r}=\sin^{-1}\left(\frac{z-H}{r}\right)$$

$$\theta=\cos^{-1}\frac{x}{R}=\sin^{-1}\frac{y}{R},$$

and taking the limit of the above as $r\to0$, it becomes $\lim_{r\to0}p(r\sin\phi,\psi)\neq0$, and

$$u^{y}_{y}=\frac{1+\nu}{8\pi(1-\nu)Er}[(3-4\nu)+\cos^2\phi\sin^2\theta]+\sum_{n=0}^{\infty}a_{n6}\frac{n!}{p^{n+1}_{n6}(r\sin\phi,\psi)},\quad r\to0.$$

The above expression is identical to the one predicted by Kelvin's solution for a unit concentrated load in the z direction. Proof of the equivalent result for all other stresses and displacements follows in a similar manner.

APPENDIX D

Two integrals which are of great importance in the previous analysis are the following:

$$\int_0^{\infty}\left[J_0\left(\lambda\frac{R}{h}\right)-\mathrm{e}^{-\lambda}\right]\frac{\mathrm{d}\lambda}{\lambda}=-\ln\frac{R}{2h}\tag{D1}$$

$$\int_0^{\infty}\left[J_0\left(\lambda\frac{R}{h}\right)-1+\frac{\lambda^2}{4}\frac{R^2}{h^2}\mathrm{e}^{-\lambda}\right]\frac{\mathrm{d}\lambda}{\lambda^3}=\frac{R^2}{4h^2}\ln\frac{R}{2h}-\frac{1}{4}\frac{R^2}{h^2}.\tag{D2}$$

Differentiating (D2) with respect to R, the following integral is obtained:

$$\int_0^{\infty}\left[J_1\left(\lambda\frac{R}{h}\right)-\frac{\lambda}{2}\frac{R}{h}\mathrm{e}^{-\lambda}\right]\frac{\mathrm{d}\lambda}{\lambda^2}=\frac{R}{4h}-\frac{R}{2h}\ln\frac{R}{2h}.\tag{D3}$$