FORCE AT A POINT IN THE INTERIOR OF A THREE-DIMENSIONAL ELASTIC LAYER

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SUMMARY

The problem considered here is that of the application of a constant force to an infinite three-dimensional, linear, elastic, isotropic, homogeneous layer. This force is assumed to be concentrated at any internal arbitrary point. The method of solution is based on the use of integral transforms. The determination of the distribution of stresses and displacements at all points of the solid are calculated in terms of convergent semi-infinite integrals.

The exact solution of the stress and displacement fields for an infinite plate is obtained when the concentrated point load is either close to or distant from the observed point. In the first case the Kelvin state solution, of the problem of a concentrated load at a point of an elastic medium occupying the entire space, is recovered. In the second case, the solution of an infinite thin plate is reproduced.

In section 3 the governing partial differential field equations, defined in section 2, are reduced to a system of ordinary differential equations by the use of the two-dimensional Fourier transform, taken with respect to the two in-plane geometric variables. Analytical expressions for the stresses and displacements are then obtained for the particular case of concentrated body forces, represented as Dirac delta functions (section 5).

Representative stress and displacement components are plotted in the last section of the paper.

1. Introduction

THE problem of a single elastic layer in equilibrium was first considered by Dougall (**1**), who conducted an extensive study of a thick plate subjected to arbitrary (surface or internal) loading using potential functions. Teodone (**2**) also dealt with this problem by using the method of mapping. Later, Orlando (**3**) obtained the solution of the layer under surface tractions.

In addition to these works, Lur'e (**4, 5**) proposed a method to construct

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particular solutions of the equations of elasticity for a layer subjected to surface loads.

Marguerre's paper (**6**) contains numerical results of the solution of the problem of a layer compressed by concentrated forces. Also, Shapiro (**7**) and Sneddon (**8**) analysed the distribution of stresses in an infinite layer for the case of normal loading, uniformly distributed over the area of a circle on the surface; besides this, Sneddon evaluated the stress field under an approximation assumption which allowed him to obtain closed-form expressions for the semi-infinite integrals presented in his work.

The object of this paper is to apply the transfer matrix formulation, used by Vlasov and Leont'ev (**9**) and generalized by Bufler (**10**), to the problem of a three-dimensional layer containing an internal concentrated unit load, which may act perpendicularly or parallel to the faces of the solid.

As an initial hypothesis no approximation is assumed except small deformations.

2. General equations

Suitable combinations of the balance law and constitutive equations of a homogeneous, isotropic linear elastic body, in component form, can be presented in the form of a matrix differential equation (**10, 11, 12**), namely

$$\frac{\partial \mathbf{a}}{\partial z} = \mathbf{A}\mathbf{a} + \mathbf{C}, \tag{2.1}$$

$$\mathbf{b} = \mathbf{B}\mathbf{a}, \tag{2.2}$$

where **a** and **b** define the column vectors

$$\mathbf{a} = (\sigma_{zz}, \sigma_{zx}, \sigma_{zy}, u_y, u_x, u_z)^T, \tag{2.3}$$

$$\mathbf{b} = (\sigma_{xx} + \sigma_{yy}, \sigma_{xx} - \sigma_{yy}, 2\sigma_{xy})^T, \tag{2.4}$$

x, y, z are spatial Cartesian coordinates in a three-dimensional Euclidean space, the superior T stands for the transpose of a vector and matrices **A** and **B** are given in (**11, 12**).

The column vector **C** is

$$\mathbf{C} = (-F_z, -F_x, -F_y, 0, 0, 0)^T,$$

where F_x, F_y, F_z stand for the body-force components.

3. Infinite layer: transformed general equations

The matrix differential equation (2.1) relates the z-coordinate partial derivative of vector **a** to the vector **a** itself. The vector **a** is composed of the components of the traction acting on a constant-z plane as well as the components of the displacements.

If x, y are the in-plane coordinates of the layer and z is the coordinate

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perpendicular to the faces, the matrix partial differential equation (2.1) can be transformed into an ordinary matrix differential equation by using the two-dimensional Fourier transform with respect to the coordinates x, y.

Thus, expressions (2.1) and (2.2) yield

$$\frac{d\bar{\mathbf{a}}}{dz} = \bar{\mathbf{A}}\bar{\mathbf{a}} + \bar{\mathbf{C}}, \tag{3.1}$$

$$\bar{\mathbf{b}} = \bar{\mathbf{B}}\bar{\mathbf{a}}, \tag{3.2}$$

where $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ and $\bar{\mathbf{C}}$ are the geometric Fourier transforms of the original vectors, defined according to Sneddon (**13**) and Bufler (**10**).

The vector $\bar{\mathbf{a}}$ will be referred to as the state vector.

4. Matrix differential equation

Equation (3.1) is an ordinary matrix differential equation which can be solved using the Cayley–Hamilton theorem (**14**).

For an arbitrary point at a distance z from the lower surface of the layer, the state vector is given by

$$\bar{\mathbf{a}}(z) = \mathbf{X}(z)\mathbf{X}^{-1}(0)\bar{\mathbf{a}}(0) + \mathbf{X}(z)\int_0^z \mathbf{X}^{-1}(s)\bar{\mathbf{C}}(s)\,ds, \tag{4.1}$$

where $\bar{\mathbf{a}}(0)$ represents the initial value of $\bar{\mathbf{a}}$ at $z = 0$, $\mathbf{X}(z)$ is the fundamental matrix defined by the matrix of eigenvectors of $\bar{\mathbf{A}}$ postmultiplied by the matrix of eigenvalues of $\bar{\mathbf{A}}$ and s is a dummy variable.

4.1. *Flexibility matrices*

Expression (4.1) can be written as

$$\bar{\mathbf{a}}(z) = \mathbf{T}(z)\bar{\mathbf{a}}(0) + \bar{\mathbf{R}}(z), \tag{4.2}$$

where

$$\mathbf{T}(z) = \mathbf{X}(z)\mathbf{X}^{-1}(0) \quad \text{and} \quad \bar{\mathbf{R}}(z) = \mathbf{X}(z)\int_0^z \mathbf{X}^{-1}(s)\bar{\mathbf{C}}(s)\,ds.$$

The state vector $\bar{\mathbf{a}}$ is composed of stresses and displacements in the following way:

$$\bar{\mathbf{a}}(z) = \begin{pmatrix} \bar{\sigma}(z) \\ \bar{u}(z) \end{pmatrix}.$$

If we are interested on relating displacements to stresses, for example stresses are known by the boundary conditions of our problem, this can be done using (4.2) in the form

$$\begin{pmatrix} \bar{\sigma}(z) \\ \bar{u}(z) \end{pmatrix} = \begin{pmatrix} \mathbf{T}_{11}(z) & \mathbf{T}_{12}(z) \\ \mathbf{T}_{21}(z) & \mathbf{T}_{22}(z) \end{pmatrix} \begin{pmatrix} \bar{\sigma}(0) \\ \bar{u}(0) \end{pmatrix} + \begin{pmatrix} \bar{\mathbf{R}}_1(z) \\ \bar{\mathbf{R}}_2(z) \end{pmatrix},$$

where $\mathbf{T}_{ij}(z)$ $(i, j = 1, 2)$ stand for the submatrices of $\mathbf{T}(z)$ and $\bar{\mathbf{R}}_i(z)$ are two

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column vectors containing the first three and the last three components of $\bar{\mathbf{R}}(z)$ respectively.

4.2. *Internal-point state vector*

For an arbitrary point, inside the layer, equation (4.2) holds. Also, for $z = h$, we obtain

$$\bar{u}(0) = -\mathbf{T}_{12}^{-1}(h)\mathbf{T}_{11}(h)\bar{\sigma}(0) + \mathbf{T}_{12}^{-1}(h)\bar{\sigma}(h) - \mathbf{T}_{12}^{-1}(h)\bar{\mathbf{R}}_1(h).$$

Substituting $\bar{u}(0)$ from above into (4.2) we get

$$\begin{pmatrix}\bar{\sigma}(z)\\ \bar{u}(z)\end{pmatrix} = \begin{pmatrix}\mathbf{T}_{11}(z) - \mathbf{T}_{12}(z)\mathbf{T}_{12}^{-1}(h)\mathbf{T}_{11}(h) & \mathbf{T}_{12}(z)\mathbf{T}_{12}^{-1}(h)\\ \mathbf{T}_{21}(z) - \mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(h)\mathbf{T}_{11}(h) & \mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(h)\end{pmatrix}\begin{pmatrix}\bar{\sigma}(0)\\ \bar{\sigma}(h)\end{pmatrix} + \\ + \begin{pmatrix}-\mathbf{T}_{12}(z)\mathbf{T}_{12}^{-1}(h)\bar{\mathbf{R}}_1(h) + \bar{\mathbf{R}}_1(z)\\ -\mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(h)\bar{\mathbf{R}}_1(h) + \bar{\mathbf{R}}_2(z)\end{pmatrix}, \tag{4.3}$$

which represents the transforms of the stresses and displacements of any arbitrary point with respect to the transforms of the tractions on the surfaces and the transform of the applied body forces.

5. Infinite layer with concentrated body forces

In this section we shall consider the solution of a layer subjected to concentrated forces of unit magnitude acting in an arbitrary direction and applied to any internal point.

Let $\xi(0, 0, H)$ be the point where the force is applied and $\mathbf{x}(x, y, z)$ be the point of observation, as depicted in Fig. 1. If $\delta(x, y, z)$ stands for the Dirac delta function defined in the geometric domain, arbitrary forces in the

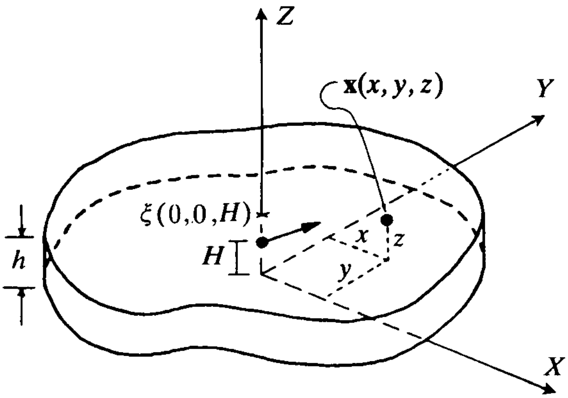

FIG. 1. Single layer with unit internal load applied to a given point. The point $\mathbf{x}$ is an arbitrary internal point

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three directions will be expressed as:

$$F^x = (\delta(x, y, z - H), 0, 0)^T,$$
$$F^y = (0, \delta(x, y, z - H), 0)^T,$$
$$F^z = (0, 0, \delta(x, y, z - H))^T,$$

where the superscript (x, y, z) denotes the direction of the unit load.

5.1. *Internal-point state vector*

In this particular case of concentrated body forces, we shall write

$$\bar{\sigma}(0) = \bar{\sigma}(h) = 0.$$

It follows from (4.3) that

$$\begin{pmatrix} \bar{\sigma}(z) \\ \bar{u}(z) \end{pmatrix} = \begin{pmatrix} -\mathbf{T}_{12}(z)\mathbf{T}_{12}^{-1}(h)\bar{\mathbf{R}}_1(h) + \bar{\mathbf{R}}_1(z)\mathcal{H}(z - H) \\ -\mathbf{T}_{22}(z)\mathbf{T}_{12}^{-1}(h)\bar{\mathbf{R}}_1(h) + \bar{\mathbf{R}}_2(z)\mathcal{H}(z - H) \end{pmatrix}, \tag{5.1}$$

with

$$\mathcal{H}(z - H) = \begin{cases} 1 & \text{for all } z \geq H; \\ 0 & \text{for all } z < H. \end{cases}$$

5.2. *Analytical expressions for the stresses and displacements*

From equation (5.1), the expressions for the stresses and displacements defined in (2.3) can be inferred. Also, making use of the transformed stresses and displacements given by (5.1) and by means of the equation (3.2), after performing the inverse transforms, the rest of the stress components will be obtained.

The analytical expressions obtained in that way are given in terms of infinite integrals. Close inspection of the expressions for the stresses σ_{zz}, σ_{zx}, σ_{zy}, σ_{xy} (when the unit load is applied along the x-, y- or z-direction), or σ_{xx} and σ_{yy} (when the unit load is applied along the x- or y-direction), demonstrates that the resulting integrands exist and are well behaved for every $\lambda \in [0, \infty)$. For $\lambda \to 0$, this was shown by expanding the integrands in ascending powers of λ and proving that the resulting expressions vanish as $\lambda \to 0$. For $\lambda \to \infty$, this was shown by replacing the hyperbolic functions involved by their equivalent exponential forms and demonstrating that the limit of the resulting expression, as $\lambda \to \infty$, vanishes.

On the other hand, analysis of the equivalent expressions for the displacements and for the stresses σ_{xx} and σ_{yy} (when the unit load is applied along the z-direction) shows that although the integrands involved are well behaved as $\lambda \to \infty$, they become singular as $\lambda \to 0$. In fact, expansion of these integrands in ascending powers of λ reveals terms of the form

$$A\left(\chi, \psi, \frac{R}{h}\right)\lambda^{-3} + B\left(\chi, \psi, \frac{R}{h}\right)\lambda^{-1}, \qquad \lambda \to 0; \tag{5.2}$$

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(A and B are known functions of $\chi = z/h$, $\psi = H/h$ and $R/h = (x^2+y^2)^{\frac{1}{2}}/h$).

The singular behaviour of the integrands indicates that the resulting expressions for the displacements are non-convergent and that the above solution should be critically reexamined.

5.3. *Proposed modifications*

The construction of the final solution to our problem was suggested by the observation that simple subtraction of terms of the form

$$A\left(\chi, \psi, \frac{R}{h}\right)\lambda^{-3} + B\left(\chi, \psi, \frac{R}{h}\right)\lambda^{-1}e^{-\lambda} \tag{5.3}$$

from the original integrands results in convergent integrals for the displacements and stresses (σ^z_{xx} and σ^z_{yy}). It should be noted here that expression (5.3) reduces to (5.2) as $\lambda \to 0$. The inclusion of the multiplying factor $e^{-\lambda}$ in the λ^{-1}-term of (5.2) ensures the integrability of the final expressions for the displacements and stresses.

It was further observed that the functions $A\lambda^{-3} + B\lambda^{-1}e^{-\lambda}$ of the transform variable λ represent Fourier transforms of displacements contributing nothing to the transforms of the stresses σ_{zz}, σ_{zx}, σ_{zy}, thus automatically satisfying the zero-traction boundary conditions at $z = h$ and $z = 0$. This is also consistent with the fact that the kernels of the integral expressions for the σ_{zz}, σ_{zx}, σ_{zy} stresses do not involve singular terms as $\lambda \to 0$.

In addition it was shown that the stress-displacement state vector corresponding to the transformed displacements $A\lambda^{-3} + B\lambda^{-1}e^{-\lambda}$ are also solutions of the transformed governing equations (3.1).

Motivated by the above observations, we propose here a solution constructed by simply subtracting singular functions of the form (5.3) from the integrands of the displacements. As mentioned above, the resulting displacements are convergent and give rise to stresses σ_{zz}, σ_{zx}, σ_{zy}, σ_{xx}, σ_{yy}, σ_{xy}, which are identical to the ones obtained directly from (5.1) and (3.2).

In the next section we shall present all displacement and stress components resulting from the modified solution. We shall then formally prove that the proposed fields satisfy all field equations and boundary conditions, and reduce to the well-known solution for a point load in an infinite domain (Kelvin state), as the point of application of the load is approached.

6. Stress and displacement fields

For the sake of simplicity only the stress and displacement fields related to a unit load in the z-direction are presented; the remaining expressions

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are provided in (**11**, **12**). Thus

$$\sigma_{zz}^{z}(x, y, z) = \frac{1}{4\pi(1-\nu)h^2}\int_{\lambda=0}^{\lambda=\infty} f_{zz}^{z}(\lambda) J_0\left(\frac{\lambda(x^2+y^2)^{\frac{1}{2}}}{h}\right) d\lambda, \tag{6.1}$$

$$\sigma_{zx}^{z}(x, y, z) = \frac{1}{4\pi(1-\nu)h^2}\frac{x}{(x^2+y^2)^{\frac{1}{2}}}\int_{\lambda=0}^{\lambda=\infty} f_{zx}^{z}(\lambda) J_1\left(\frac{\lambda(x^2+y^2)^{\frac{1}{2}}}{h}\right) d\lambda, \tag{6.2}$$

$$\sigma_{zy}^{z}(x, y, z) = \frac{1}{4\pi(1-\nu)h^2}\frac{y}{(x^2+y^2)^{\frac{1}{2}}}\int_{\lambda=0}^{\lambda=\infty} f_{zy}^{z}(\lambda) J_1\left(\frac{\lambda(x^2+y^2)^{\frac{1}{2}}}{h}\right) d\lambda, \tag{6.3}$$

$$\begin{aligned}\sigma_{xx}^{z}(x, y, z) = \frac{\nu}{1-\nu}\sigma_{zz}^{z}(x, y, z) + \frac{1}{8\pi(1-\nu)^2h^2}\Bigg\{(1+\nu)\int_{\lambda=0}^{\lambda=\infty}\lambda\, \times \\ \times\left[f_x^z(\lambda) J_0\left(\frac{\lambda(x^2+y^2)^{\frac{1}{2}}}{h}\right) - 12(1-\nu)^2(2\chi-1)\frac{e^{-\lambda}}{\lambda}\right] d\lambda\, + \\ + (1-\nu)\frac{y^2-x^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda f_x^z(\lambda) J_2\left(\frac{\lambda(x^2+y^2)^{\frac{1}{2}}}{h}\right) d\lambda\Bigg\},\end{aligned} \tag{6.4}$$

$$\begin{aligned}\sigma_{yy}^{z}(x, y, z) = \frac{\nu}{1-\nu}\sigma_{zz}^{z}(x, y, z) + \frac{1}{8\pi(1-\nu)^2h^2}\Bigg\{(1+\nu)\int_{\lambda=0}^{\lambda=\infty}\lambda\, \times \\ \times\left[f_y^z(\lambda) J_0\left(\frac{\lambda(x^2+y^2)^{\frac{1}{2}}}{h}\right) - 12(1-\nu)^2(2\chi-1)\frac{e^{-\lambda}}{\lambda}\right] d\lambda\, + \\ + (1-\nu)\frac{x^2-y^2}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda f_y^z(\lambda) J_2\left(\frac{\lambda(x^2+y^2)^{\frac{1}{2}}}{h}\right) d\lambda\Bigg\},\end{aligned} \tag{6.5}$$

$$\sigma_{xy}^{z}(x, y, z) = -\frac{1}{4\pi(1-\nu)h^2}\frac{xy}{x^2+y^2}\int_{\lambda=0}^{\lambda=\infty}\lambda f_y^z(\lambda) J_2\left(\frac{\lambda(x^2+y^2)^{\frac{1}{2}}}{h}\right) d\lambda, \tag{6.6}$$

$$\begin{aligned}u_z^z(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\int_{\lambda=0}^{\lambda=\infty}\Bigg\{f_z^z(\lambda) J_0\left(\frac{\lambda(x^2+y^2)^{\frac{1}{2}}}{h}\right) - \frac{24(1-\nu)^2}{\lambda^3} \\ -\Bigg[12\nu(1-\nu)(\chi+\psi-\chi^2-\psi^2) + \tfrac{24}{5}(1-\nu)^2 \\ -6(1-\nu)^2\frac{x^2+y^2}{h^2}\Bigg]\frac{e^{-\lambda}}{\lambda}\Bigg\} d\lambda,\end{aligned} \tag{6.7}$$

$$\begin{aligned}u_x^z(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{x}{(x^2+y^2)^{\frac{1}{2}}}\int_{\lambda=0}^{\lambda=\infty}\Bigg\{f_x^z(\lambda) J_1\left(\frac{\lambda(x^2+y^2)^{\frac{1}{2}}}{h}\right) \\ -6(1-\nu)^2\frac{(x^2+y^2)^{\frac{1}{2}}}{h}(2\chi-1)\frac{e^{-\lambda}}{\lambda}\Bigg\} d\lambda,\end{aligned} \tag{6.8}$$

$$\begin{aligned}u_y^z(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)h}\frac{1}{E}\frac{y}{(x^2+y^2)^{\frac{1}{2}}}\int_{\lambda=0}^{\lambda=\infty}\Bigg\{f_y^z(\lambda) J_1\left(\frac{\lambda(x^2+y^2)^{\frac{1}{2}}}{h}\right) \\ -6(1-\nu)^2\frac{(x^2+y^2)^{\frac{1}{2}}}{h}(2\chi-1)\frac{e^{-\lambda}}{\lambda}\Bigg\} d\lambda.\end{aligned} \tag{6.9}$$

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Here the superscript (x, y, z) indicates the direction of the unit load, and $\chi = z/h$, $\psi = H/h$; the $f(\lambda)$ are functions of λ, χ and ψ and are defined in (**11**, **12**).

6.1. *Basic features of the solution*

In a former paper (**12**), we discussed the characteristic features of the solution presented above; see expressions (6.1) to (6.9). In particular the following properties were demonstrated.

(1) The expressions for the displacement field satisfy the displacement equations of equilibrium.

(2) The proposed stress field satisfies the boundary conditions prescribed on the plate surfaces.

(3) The integral of the tractions over the boundary ∂C of a cylinder of arbitrary radius ρ, $\rho > 0$ is equal to minus the point load applied at $\xi = He_z$.

(4) The stress and displacement fields have the following properties:

$$\left.\begin{aligned} \sigma(x) &= o(r^{-2}) \\ u(x) &= o(r^{-1}) \end{aligned}\right\} \quad r \to 0, \qquad r = (x^2 + y^2 + (z - H)^2)^{\frac{1}{2}} > 0.$$

In particular, the displacements and stresses of the present solution reduce to the equivalent ones predicted by Kelvin's solution as the point of application of the load is approached. The proof of the above for the specific case of a stress component is given in (**12**).

The complete proof for all displacement and stress components for concentrated loads along any direction is entirely analogous.

6.2. *Far-field analysis of stress and displacement components*

The solution defined by expressions given in section 6 is formally satisfactory. However, for some of the expressions, a further integration would lend them an easier physical interpretation.

In particular, the solution in its present form throws no light on the question of the behaviour at points which are a large distance from the applied load in comparison with the plate thickness. By using the integrals defined in the Appendix the expressions, from which singular terms have been subtracted, (6.10), (6.11), (6.19) to (6.23), (6.25), (6.27) in (**11**, **12**), are shown to be composed of two parts of very different characters. The first part is a function the value of which decreases as the distance from the source increases, while the second part is a function of a very simple form. Thus, the solution is separated into a local, transitory, or decaying part, which fades away from the neighbourhood of the applied load; and a permanent, or persistent part, which is important in the whole domain

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occupied by the layer. Now

$$\sigma^z_{xx}(x, y, z) = \frac{\nu}{1-\nu}\sigma^z_{zz}(x, y, z) + \\ + \frac{1}{8\pi(1-\nu)^2h^2}\left\{(1+\nu)\int_{\lambda=0}^{\lambda=\infty}\left[\lambda f^z_x(\lambda) - 12(1-\nu)^2(2\chi-1)\frac{1}{\lambda}\right]\times \right. \\ \left. \times J_0\left(\lambda\frac{R}{h}\right)d\lambda + (1-\nu)\frac{y^2-x^2}{R^2}\int_{\lambda=0}^{\lambda=\infty}\lambda f^z_x(\lambda)J_2\left(\lambda\frac{R}{h}\right)d\lambda\right\} \\ - \frac{3(1+\nu)}{2\pi h^2}(2\chi-1)\ln\frac{R}{2h}, \tag{6.10}$$

$$\sigma^z_{yy}(x, y, z) = \frac{\nu}{1-\nu}\sigma^z_{zz}(x, y, z) + \\ + \frac{1}{8\pi(1-\nu)^2h^2}\left\{(1+\nu)\int_{\lambda=0}^{\lambda=\infty}\left[\lambda f^z_y(\lambda) - 12(1-\nu)^2(2\chi-1)\frac{1}{\lambda}\right]\times \right. \\ \left. \times J_0\left(\lambda\frac{R}{h}\right)d\lambda + (1-\nu)\frac{x^2-y^2}{R^2}\int_{\lambda=0}^{\lambda=\infty}\lambda f^z_y(\lambda)J_2\left(\lambda\frac{R}{h}\right)d\lambda\right\} \\ - \frac{3(1+\nu)}{2\pi h^2}(2\chi-1)\ln\frac{R}{2h}, \tag{6.11}$$

$$u^z_z(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)hE}\int_{\lambda=0}^{\lambda=\infty}\left\{f^z_z(\lambda) - \frac{24(1-\nu)^2}{\lambda^3}\right. \\ \left. - [12\nu(1-\nu)(\chi+\psi-\chi^2-\psi^2) + \tfrac{24}{5}(1-\nu)^2]\frac{1}{\lambda}\right\}J_0\left(\lambda\frac{R}{h}\right)d\lambda + \\ + \frac{1+\nu}{4\pi(1-\nu)hE}\left\{\ln\frac{R}{2h}\left[6(1-\nu)^2\frac{R^2}{h^2}\right.\right. \\ \left.\left. - 12\nu(1-\nu)(\chi+\psi-\chi^2-\psi^2) - \tfrac{24}{5}(1-\nu)^2\right] - 6(1-\nu)^2\frac{R^2}{h^2}\right\}, \tag{6.12}$$

$$u^z_x(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)hE}\frac{x}{R}\int_{\lambda=0}^{\lambda=\infty}\left\{f^z_x(\lambda) - 12(1-\nu)^2(2\chi-1)\frac{1}{\lambda^2}\right\}J_1\left(\lambda\frac{R}{h}\right)d\lambda + \\ + \frac{3(1-\nu^2)}{2\pi hE}(2\chi-1)\frac{x}{R}\left[\frac{R}{2h} - \frac{R}{h}\ln\frac{R}{2h}\right], \tag{6.13}$$

$$u^z_y(x, y, z) = \frac{1+\nu}{4\pi(1-\nu)hE}\frac{y}{R}\int_{\lambda=0}^{\lambda=\infty}\left\{f^z_y(\lambda) - 12(1-\nu)^2(2\chi-1)\frac{1}{\lambda^2}\right\}J_1\left(\lambda\frac{R}{h}\right)d\lambda + \\ + \frac{3(1-\nu^2)}{2\pi hE}(2\chi-1)\frac{y}{R}\left[\frac{R}{2h} - \frac{R}{h}\ln\frac{R}{2h}\right], \tag{6.14}$$

where $R = (x^2+y^2)^{\frac{1}{2}}$.

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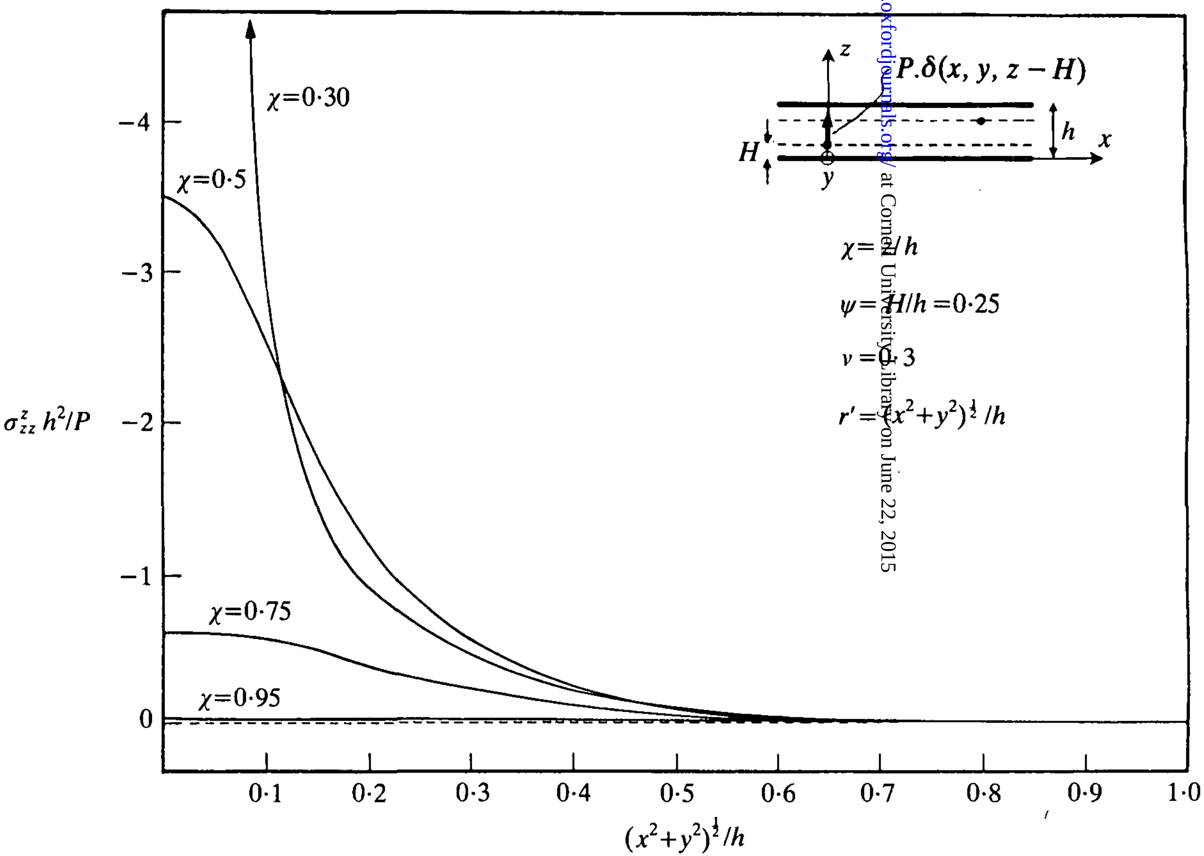

FIG. 2. Variation of the normalized stress $\sigma^z_{zz}h/P$ versus the normalized in-plane distance $r' = (x^2+y^2)^{\frac{1}{2}}/h$

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In all of the previous components, the integrands of the transitory parts exist and are well behaved for every $\lambda \in [0, \infty)$. And, in particular, the limit of these integrands is zero as $\lambda \to 0$ and $\lambda \to \infty$.

6.3. *Numerical evaluation of the solution*

Examples demonstrating some of the features of the three-dimensional solution are presented in Figs 2, 3 and 4. A point load along the z-direction was applied at a distance $0{\cdot}25h$ from the lower surface of the layer. The variation of the σ^z_{zz}-component of the stresses with respect to the normalized in-plane distance $r^1 = (x^2 + y^2)^{\frac{1}{2}}/h$ measured from the point of application of the load is shown for the cases of $z = 0{\cdot}95h$, $z = 0{\cdot}75h$, $z = 0{\cdot}5h$ and $z = 0{\cdot}3h$. As expected, as $r \to 0$ ($z \to 0{\cdot}25h$, $r' \to 0$), the stresses reproduce the singular behaviour of the Kelvin state.

Figures 3 and 4 show the variation of the same stress component along the thickness of the plate for different values of the normalized in-plane distance $(x^2 + y^2)^{\frac{1}{2}}/h$ measured from the applied load. At distances close to

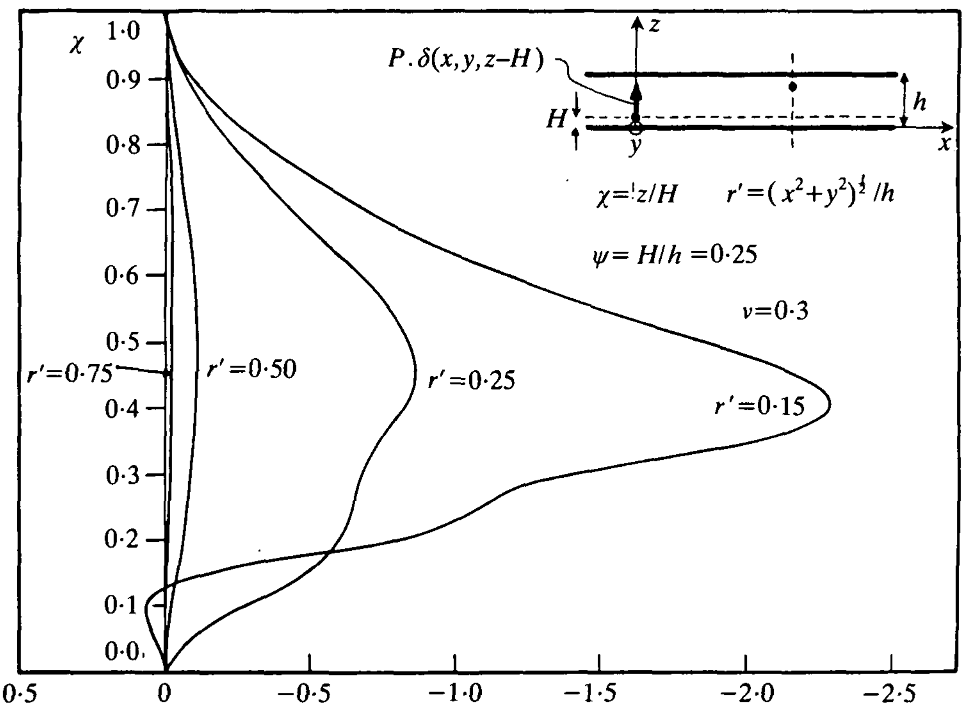

FIG. 3. Variation of the normalized stress $\sigma^z_{zz}h^2/P$ versus the normalized distance $\chi = z/h$ measured from the lower surface of the layer. Different curves correspond to $r' = 0{\cdot}75$, $r' = 0{\cdot}50$, $r' = 0{\cdot}25$, $r' = 0{\cdot}15$

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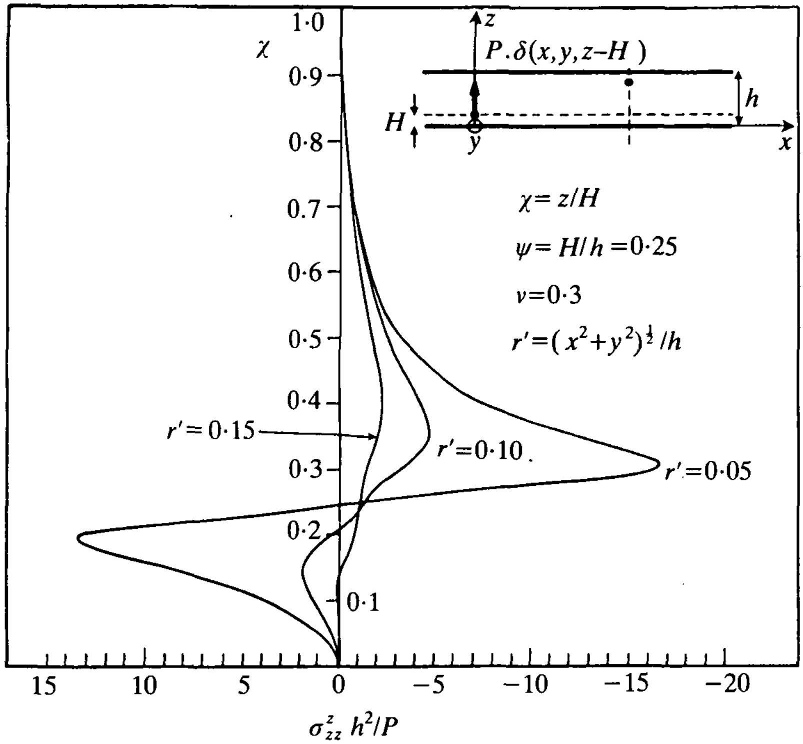

FIG. 4. Variation of the normalized stress $\sigma^z_{zz}h^2/P$ versus the normalized distance $\chi = z/h$ measured from the lower surface of the layer. Different curves correspond to $r' = 0{\cdot}15$, $r' = 0{\cdot}10$, $r' = 0{\cdot}05$

the load (see Fig. 4, $(x^2+y^2)^{\frac{1}{2}}/h = 0{\cdot}05$), the stress changes rapidly from tensile to compressive as the plane of application of the load ($z = 0{\cdot}25h$) is traversed. As the distance from the load is increased, the tensile portion of the thickness variation diminishes and eventually disappears. It is also worth noting that for distances greater than $0{\cdot}5h$ the thickness variation becomes symmetrically shaped despite the fact that the problem is non-symmetric in the thickness direction, suggesting that the decay length for the three-dimensional Saint-Venant problem is of the order of half the plate thickness.

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APPENDIX

Two integrals of great importance in the above analysis are the following:

$$\int_0^\infty \left[J_0\left(\lambda\frac{R}{h}\right)-e^{-\lambda}\right]\frac{d\lambda}{\lambda}=-\ln\frac{R}{2h}, \tag{A.1}$$

$$\int_0^\infty \left[J_0\left(\lambda\frac{R}{h}\right)-1+\frac{\lambda^2}{4}\frac{R^2}{h^2}e^{-\lambda}\right].\frac{d\lambda}{\lambda^3}=\frac{R^2}{4h^2}\ln\frac{R}{2h}-\frac{1}{4}\frac{R^2}{h^2}. \tag{A.2}$$

Differentiating (A.2) with respect to R, the following integral is also obtained:

$$\int_0^\infty \left[J_1\left(\lambda\frac{R}{h}\right)-\frac{\lambda}{2}\frac{R}{h}e^{-\lambda}\right]\frac{d\lambda}{\lambda^2}=\frac{R}{4h}-\frac{R}{2h}\ln\frac{R}{2h}. \tag{A.3}$$

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