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Negative plastic flow and its prevention in elasto-plastic finite element computation

Xiaomin Deng * and Ares J. Rosakis
California Institute of Technology, Pasadena, CA 91125, U.S.A.

Abstract. The phenomenon of negative plastic flow and its prevention in elasto-plastic numerical analyses, such as finite element computations, are investigated in detail. Discussions are confined to materials with smooth but otherwise arbitrary initial and post yield surfaces which obey associated or non-associated flow rules, with specializations for materials obeying the von Mises yield condition and its associated flow rule, and to materials with cornered yield surfaces obeying the associated flow rule. Current ad hoc treatments of this phenomenon and their effects on the accuracy and reliability of computer codes are discussed. Explicit results for some commonly used stress integration algorithms are presented as to when and why negative plastic flow will occur, and how it can be avoided with proper modification to the existing stress calculation procedures. Meanwhile, it is pointed out that in the case negative flow happens to tangent stiffness based algorithms, numerical errors are also involved in other algorithms. Finally, a general and numerically efficient approach for stress calculation at corner yield points is proposed, which is consistent with the Koiter's flow rule. The subject of uniqueness of the Koiter's flow rule representation is addressed in detail.

Introduction

Elasto-plastic finite element computation is a common practice both in scientific research and in engineering applications. Due to accuracy restrictions often required and due to significant consumptions of computer resources involved in such nonlinear analyses, it is of vital importance that reliable as well as efficient solution algorithms be employed. A key factor which affects the reliability of a computer program is the appearance of negative plastic flow and its related effects, which will become clear in later analysis, especially in cases where complex loading and/or large loading increments are present.

In the classic incremental theory of plasticity, the plastic strain increment proportionality coefficient, $d\lambda$, which is usually called the plastic multiplier, or the flow factor, and will be defined later, is required to be non-negative when the stress state is actively yielded. Numerically, when it takes negative values, it is said that negative plastic flow has occurred. However, although it has long been known that negative plastic flow causes deterioration of numerical solutions with oscillations and even divergence, it seems that this subject has not been treated formally and a clear understanding of this phenomenon has not been achieved.

Negative plastic flow has been encountered since the beginning of elasto-plastic finite element computations. Marcal [7], in applying his pioneering Tangent Stiffness method, stated that it was not possible for him to ensure that $d\lambda$ always remained non-negative. Hence he proposed to check the value of $d\lambda$ and to stop the computation when a negative value was encountered [7,9]. This rule treats negative plastic flow as the appearance of unacceptable numerical errors, and was followed, for example, by Yamada and Yoshimura [19], who

* Present address: Department of Mechanical Engineering, University of South Carolina, Columbia, SC 29208, U.S.A.

recognized that the implication of $d\lambda < 0$ was not fully understood. With such a rule, the capability of numerical elasto-plastic computations is greatly hindered.

However, continued computation is usually desired when a negative $d\lambda$ value is indeed detected. In order to do so, an ad hoc remedy is commonly employed. In such a case, negative $d\lambda$ values are simply replaced with zeros [11,13]. Yet, the effect of this zeroing of the negative $d\lambda$ values on the accuracy and reliability of a computer program is never studied.

A possibility exists that no check is placed on the sign of $d\lambda$ in a computer program, which is usually the case when only the stress and strain states at the end of a loading increment are of interest and all the intermediate steps are omitted. In such circumstances, the phenomenon of negative plastic flow will pass unnoticed. Likewise, unpredictable results may occur unnoticed.

In this paper we report the results of a detailed investigation into the phenomenon of negative plastic flow and some relevant issues for displacement based elasto-plastic finite element methods. Explicit conclusions are presented as to when negative plastic flow occurs and why it occurs. It is pointed out that while negative plastic flow may never occur with certain stress integration algorithms, the flawed solution procedures in wide use today will still predict wrong results using those algorithms, in circumstances where $d\lambda < 0$ does appear with the classic tangent stiffness algorithm. Moreover, stress integration algorithms for von Mises materials, such as the Tangent Stiffness method and its modified versions, the Secant Stiffness method and the Radial Return method, are discussed.

In our discussion, attention is especially given to cases where yield surfaces with corners are employed. A general and numerically efficient approach for stress calculation at corner yield points is provided, which is consistent with the Koiter's flow rule [5]. The subject of uniqueness of the Koiter's flow rule representation is addressed in detail.

Incremental theory of plasticity

Consider an elastic–plastic solid which obeys the following yield condition:

$$F(\sigma, \epsilon^p, k) \equiv f(\sigma, \epsilon^p) - h(k) = 0, \qquad (1)$$

where F is the yield function composed of f, a convex function in the stress space of the stress tensor σ and the plastic strain tensor ϵ^p, and h, a non-decreasing function of the hardening parameter k. The functions f and h respectively define the shape and the size of the yield surface given by (1) in the stress space. Note that the yield surface is convex in the stress space.

Following Nayak and Zienkiewicz [11], we suppose that there exists a plastic potential function

$$Q(\sigma, \epsilon^p, k_0) \equiv q(\sigma, \epsilon^p) - l(k_0) = 0, \qquad (2)$$

such that the normality principle is expressed according to the flow rule as

$$d\epsilon^p = d\lambda\, \hat{a}, \qquad (3)$$

where

$$\hat{a} = \frac{\partial Q}{\partial \sigma} = \frac{\partial q}{\partial \sigma} \qquad (4)$$

is, in the stress space, the gradient tensor of both Q and q, and is normal to the hypersurface denoted by (2), and $d\lambda$ is the plastic strain increment proportionality coefficient, or simply the plastic multiplier or the flow factor, and it takes zero value for an elastic stress state and is non-negative for a currently yielded stress state. The relation (3) is called an associated flow rule if $Q = F$, otherwise it is a non-associated flow rule.

Now the Prandtl–Reuss strain increment decomposition assumption states that

$$\mathrm{d}\boldsymbol{\epsilon} = \mathrm{d}\boldsymbol{\epsilon}^{\mathrm{e}} + \mathrm{d}\boldsymbol{\epsilon}^{\mathrm{p}}, \tag{5}$$

where $\mathrm{d}\boldsymbol{\epsilon}^{\mathrm{e}}$ and $\mathrm{d}\boldsymbol{\epsilon}^{\mathrm{p}}$ are respectively the infinitesimal elastic and plastic strain increment tensor, with $\mathrm{d}\boldsymbol{\epsilon}^{\mathrm{e}}$ related to the infinitesimal stress increment tensor $\mathrm{d}\boldsymbol{\sigma}$ through

$$\mathrm{d}\boldsymbol{\sigma} = \boldsymbol{C}\,\mathrm{d}\boldsymbol{\epsilon}^{\mathrm{e}}, \tag{6}$$

where $\boldsymbol{C}$ is the fourth-order elasticity modulus tensor of the solid, with the usual primary and secondary symmetries. It is assumed here that $\boldsymbol{C}$ is positive definite, but need not be homogeneous and isotropic, unless otherwise stated.

When the yield condition (1) is used in its incremental form, we obtain the consistency condition

$$\mathrm{d}F = \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{\sigma} - A\,\mathrm{d}\lambda = 0, \tag{7}$$

where $\cdot$ is the tensor inner product operator; $\boldsymbol{a}$ defined by

$$\boldsymbol{a} = \frac{\partial F}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{8}$$

is, in the stress space, the gradient tensor of both F and f, and is normal to the hypersurface denoted by (1); A is a material parameter defining the subsequent yield surfaces and is given by

$$A = -\mathrm{d}\lambda^{-1}\left(\frac{\partial F}{\partial \boldsymbol{\epsilon}^{\mathrm{p}}}\cdot\mathrm{d}\boldsymbol{\epsilon}^{\mathrm{p}} + \frac{\partial F}{\partial k}\mathrm{d}k\right) = \mathrm{d}\lambda^{-1}\left(-\frac{\partial f}{\partial \boldsymbol{\epsilon}^{\mathrm{p}}}\cdot\mathrm{d}\boldsymbol{\epsilon}^{\mathrm{p}} + \frac{\mathrm{d}h}{\mathrm{d}k}\mathrm{d}k\right). \tag{9}$$

From (3)–(7), the plastic multiplier can be obtained as

$$\mathrm{d}\lambda = \frac{\boldsymbol{a}\cdot(\boldsymbol{C}\,\mathrm{d}\boldsymbol{\epsilon})}{A + \boldsymbol{a}\cdot(\boldsymbol{C}\hat{\boldsymbol{a}})}. \tag{10}$$

Substituting $\mathrm{d}\lambda$ into (3) and rearranging (6) with the help of (3) and (5), we obtain:

$$\mathrm{d}\boldsymbol{\sigma} = \boldsymbol{C}^{\mathrm{ep}}\,\mathrm{d}\boldsymbol{\epsilon}, \tag{11}$$

where $\boldsymbol{C}^{\mathrm{ep}}$ is the fourth-order elastic–plastic tangent stiffness tensor defined by

$$\boldsymbol{C}^{\mathrm{ep}} = \boldsymbol{C} - \frac{(\boldsymbol{C}\hat{\boldsymbol{a}})\otimes(\boldsymbol{C}\boldsymbol{a})}{A + \boldsymbol{a}\cdot(\boldsymbol{C}\hat{\boldsymbol{a}})}, \tag{12}$$

with $\otimes$ being the tensor outer product operator.

Equation (11) represents the well-known classic small strain incremental stress–strain relationship. It is the linearized representation of the non-linear incremental plasticity theory described by equations (1), (3), (5) and (6). We will call the numerical stress computation algorithms directly based on (11) the tangent stiffness based algorithms. On the other hand, when a numerical algorithm is based on the set of non-linear elastic–plastic equations or their modified versions, it is called an implicit algorithm [18].

In the following sections it is assumed that the material parameter A is positive for hardening solids, and is zero for non-hardening solids, and that the yield function and the plastic potential function are such that the inner product of their gradients, $\boldsymbol{a}$ and $\hat{\boldsymbol{a}}$, in the stress space with respect to the elasticity tensor $\boldsymbol{C}$ is positive.

Elasto-plastic computation for smooth yield surfaces

Stress calculation procedure

The following procedure is commonly adopted for stress computations for materials with smooth yield and plastic potential surfaces, which are defined here as those whose gradient tensors, $\boldsymbol{a}$ and $\hat{\boldsymbol{a}}$, respectively, are everywhere uniquely defined. Now consider a stress state $\boldsymbol{\sigma}$ with a finite strain increment $\Delta\boldsymbol{\epsilon}$. In principle, the new stress state is computed as follows.

First, assuming elastic behavior, the trial stress increment is calculated as

$$\Delta\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{C}\,\Delta\boldsymbol{\epsilon}, \tag{13}$$

and a trial stress state is obtained through

$$\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}^{\mathrm{T}}. \tag{14}$$

The trial stress $\boldsymbol{\sigma}^{\mathrm{T}}$ is then tested in $F(\boldsymbol{\sigma}^{\mathrm{T}}, \boldsymbol{\epsilon}^{\mathrm{p}}, k)$. If $F < 0$, then the elasticity assumption is taken to be valid, and $\boldsymbol{\sigma}^{\mathrm{T}}$ is considered as the new stress state. Otherwise, the strain increment is partly in an elastic path and partly in a plastic path, with the transition point, usually denoted by a scalar parameter $R \in [0, 1]$, being determined through

$$F(\boldsymbol{\sigma} + R\Delta\boldsymbol{\sigma}^{\mathrm{T}}, \boldsymbol{\epsilon}^{\mathrm{p}}, k) = 0. \tag{15}$$

Then the stress state, defined by

$$\boldsymbol{\sigma}^{\mathrm{C}} = \boldsymbol{\sigma} + R\Delta\boldsymbol{\sigma}^{\mathrm{T}}, \tag{16}$$

is calculated, which is called the contact stress and lies exactly on the current yield surface. Note that such solution procedures differ only in the way the transition parameter R is determined.

Once the contact stress is determined, the remaining plastic computations all will be referred to it. From (10) the finite flow factor increment will be given by

$$\Delta\lambda = \frac{(1-R)\boldsymbol{a}\cdot(\boldsymbol{C}\,\Delta\boldsymbol{\epsilon})}{A + \boldsymbol{a}\cdot(\boldsymbol{C}\hat{\boldsymbol{a}})} = \frac{(1-R)\boldsymbol{a}\cdot\Delta\boldsymbol{\sigma}^{\mathrm{T}}}{A + \boldsymbol{a}\cdot(\boldsymbol{C}\hat{\boldsymbol{a}})}. \tag{17}$$

Finally, the new stress state is obtained through

$$\boldsymbol{\sigma}^{\mathrm{N}} = \boldsymbol{\sigma}^{\mathrm{C}} + \Delta\boldsymbol{\sigma}, \tag{18}$$

where $\Delta\boldsymbol{\sigma}$ can be computed from

$$\Delta\boldsymbol{\sigma} = (1-R)\boldsymbol{C}^{\mathrm{ep}}\,\Delta\boldsymbol{\epsilon}, \tag{19}$$

or more efficiently from

$$\Delta\boldsymbol{\sigma} = (1-R)\;\Delta\boldsymbol{\sigma}^{\mathrm{T}} - \Delta\lambda\boldsymbol{C}\hat{\boldsymbol{a}}. \tag{20}$$

For clarity of discussion, more elaborations regarding stress calculation, such as final stress corrections for the above stress estimate, subincrementation of the strain increment, and path-independency for iterative finite element formulations, will not be described here, but can be found in references by Nayak and Zienkiewicz [11], by Schreyer, Kulak and Kramer [15], by Owen and Hinton [13], by Marques [9], by Bathe, Ramm and Wilson [2], by Mondkar and Powell [10], and by Nyssen [12]. Nevertheless, omission of those elaborations will not affect our discussion.

From (17) it is immediately seen that negative plastic flow will occur for the above tangent stiffness algorithm when the inner product between $\boldsymbol{a}$ and $\Delta\boldsymbol{\sigma}^{\mathrm{T}}$ is negative, since $(1-R)$ is by definition non-negative and the denominator is assumed to be positive. Suppose $\boldsymbol{n}$ is the outward unit normal to the current yield surface (1) in the stress space at the stress point $\boldsymbol{\sigma}^{\mathrm{C}}$, then $\boldsymbol{a}$, the gradient of the yield surface at that point, is by definition along the direction of $\boldsymbol{n}$. Hence, equivalently, $\Delta\lambda$ takes a negative value when the projection of $\Delta\boldsymbol{\sigma}^{\mathrm{T}}$ onto $\boldsymbol{n}$ is negative, i.e. when the end point of $\Delta\boldsymbol{\sigma}^{\mathrm{T}}$, which begins at the stress point $\boldsymbol{\sigma}^{\mathrm{C}}$, is below the local tangent plane of the yield surface at that point. Since the yield surface is convex everywhere, and the strain path, and hence the trial stress increment, are numerically linear during each strain increment, the above situation will occur only when $\boldsymbol{\sigma}$, the stress state at the beginning of the

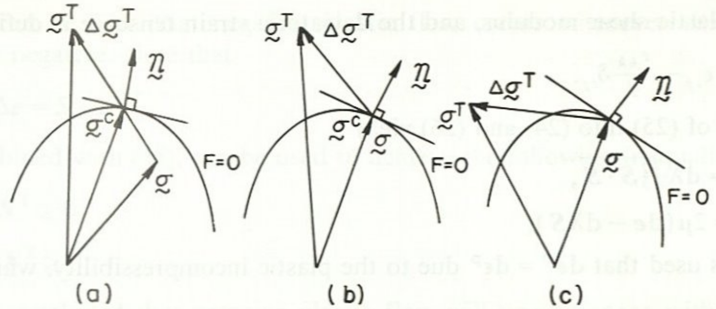

Fig. 1. Geometrical representations of the yield surface $F=0$, its outward normal $\boldsymbol{n}$ and its tangent plane at the calculated contact stress state $\boldsymbol{\sigma}^{\mathrm{C}}$, the initial stress state $\boldsymbol{\sigma}$, the trial stress increment $\Delta\boldsymbol{\sigma}^{\mathrm{T}}$, and the trial stress state $\boldsymbol{\sigma}^{\mathrm{T}}$: (a) initial stress state is inside the yield surface; (b) initial stress state is on the yield surface and the trial stress increment stays outside; (c) initial stress state is on the yield surface and the trial stress increment intercepts the yield surface.

increment, is previously yielded and is taken to be the contact stress state $\boldsymbol{\sigma}^{\mathrm{C}}$, and when the stress state is first elastically unloaded and then reloaded to yield again. Graphically, such situations are schematically illustrated in Fig. 1.

Specialization for J_2 flow theory of plasticity

For implicit algorithms, whether negative plastic flow ever will occur, depends very much on the way the flow rule (3) is approximated, which will not be discussed here. But the effect of the approximation of (3) can be observed from the use of some popular algorithms for plastic stress calculation for isotopic materials obeying the J_2 flow theory of plasticity, as described below.

In this theory, the von Mises yield condition,

$$F(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^{\mathrm{p}}, k) \equiv J_2(\boldsymbol{S}) - \tfrac{1}{3}\sigma^2(\epsilon_{\mathrm{p}}) = 0, \tag{21}$$

is employed, where $\boldsymbol{S}$ and J_2 are, respectively, the deviatoric stress tensor and its second invariant, defined respectively by

$$S_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}, \tag{22}$$

$$J_2 = \tfrac{1}{2}\boldsymbol{S}\cdot\boldsymbol{S}, \tag{23}$$

σ is the yield stress in tension and is usually called the flow stress, and ϵ_{p} is the effective plastic strain given by

$$\epsilon_{\mathrm{p}} = \int \sqrt{\tfrac{2}{3}\dot{\boldsymbol{\epsilon}}^{\mathrm{p}}\cdot\dot{\boldsymbol{\epsilon}}^{\mathrm{p}}}\ \mathrm{d}t. \tag{24}$$

The dependence of the flow stress σ on ϵ_{p} is called the hardening rule which describes the hardening behavior of the material. If σ is a constant, then it models an elastic–perfectly plastic solid.

From the above relations, the associated flow rule can be written as

$$\mathrm{d}\boldsymbol{\epsilon}^{\mathrm{p}} = \mathrm{d}\lambda \boldsymbol{S}, \tag{25}$$

and the elastic response of the material can be described in terms of the deviatoric stress and strain tensors as

$$\mathrm{d}\boldsymbol{S} = 2\mu(\mathrm{d}\boldsymbol{e} - \mathrm{d}\boldsymbol{e}^{\mathrm{p}}), \tag{26}$$

where μ is the elastic shear modulus, and the deviatoric strain tensor $\boldsymbol{e}$ is defined as

$$e_{ij} = \epsilon_{ij} - \frac{\epsilon_{kk}}{3}\delta_{ij}. \tag{27}$$

Substitutions of (25) into (24) and (26) yield

$$\mathrm{d}\epsilon_{\mathrm{p}} = \mathrm{d}\lambda\sqrt{\tfrac{2}{3}\boldsymbol{S}\cdot\boldsymbol{S}}\,, \tag{28}$$

$$\mathrm{d}\boldsymbol{S} = 2\mu(\mathrm{d}\boldsymbol{e} - \mathrm{d}\lambda\boldsymbol{S}), \tag{29}$$

where the fact is used that $\mathrm{d}\boldsymbol{e}^{\mathrm{p}} = \mathrm{d}\boldsymbol{\epsilon}^{\mathrm{p}}$ due to the plastic incompressibility, which is implied by (25) and (22).

Moreover, with the use of (28), the consistency condition during plastic flow can be obtained in incremental form from the yield condition as

$$\boldsymbol{S}\cdot\mathrm{d}\boldsymbol{S} = \frac{2}{3}\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon_{\mathrm{p}}}\boldsymbol{S}\cdot\boldsymbol{S}\,\mathrm{d}\lambda. \tag{30}$$

When use is made of (29) and (30), the following formula for the plastic flow factor can be obtained:

$$\mathrm{d}\lambda = \frac{2\mu\boldsymbol{S}\cdot\mathrm{d}\boldsymbol{e}}{\boldsymbol{S}\cdot\boldsymbol{S}\left(\dfrac{2}{3}\dfrac{\mathrm{d}\sigma}{\mathrm{d}\epsilon_{\mathrm{p}}} + 2\mu\right)}, \tag{31}$$

whose substitution into (29) produces the incremental deviatoric stress–strain relation:

$$\mathrm{d}\boldsymbol{S} = 2\mu\left[\boldsymbol{I} - \frac{2\mu\boldsymbol{S}\otimes\boldsymbol{S}}{\boldsymbol{S}\cdot\boldsymbol{S}\left(\dfrac{2}{3}\dfrac{\mathrm{d}\sigma}{\mathrm{d}\epsilon_{\mathrm{p}}} + 2\mu\right)}\right]\mathrm{d}\boldsymbol{e}, \tag{32}$$

where $\boldsymbol{I}$ is the fourth-order identity tensor. The last two equations are the basis for stress calculation during plastic flow.

Now suppose in a finite element computation that the contact stress tensor $\boldsymbol{S}^{\mathrm{C}}$ and the remaining finite strain increment tensor $\Delta\boldsymbol{e}$ have been determined. For convenience, we shall call $\Delta\boldsymbol{S}^{\mathrm{T}} = 2\mu\Delta\boldsymbol{e}$ the trial stress increment tensor, $\boldsymbol{S}^{\mathrm{T}} = \boldsymbol{S}^{\mathrm{C}} + \Delta\boldsymbol{S}^{\mathrm{T}}$ the trial stress tensor, and $\boldsymbol{S}^{\mathrm{M}} = \frac{1}{2}(\boldsymbol{S}^{\mathrm{C}} + \boldsymbol{S}^{\mathrm{T}}) = \boldsymbol{S}^{\mathrm{C}} + \frac{1}{2}\Delta\boldsymbol{S}^{\mathrm{T}}$ the mean stress tensor. Then, by definition, we have for the yield function, with the current flow stress σ:

$$\tfrac{1}{2}\boldsymbol{S}^{\mathrm{C}}\cdot\boldsymbol{S}^{\mathrm{C}} - \tfrac{1}{3}\sigma^2 = 0, \tag{33}$$

$$\tfrac{1}{2}\boldsymbol{S}^{\mathrm{T}}\cdot\boldsymbol{S}^{\mathrm{T}} - \tfrac{1}{3}\sigma^2 \geq 0, \tag{34}$$

which can be used to derive

$$\boldsymbol{S}^{\mathrm{C}}\cdot\Delta\boldsymbol{S}^{\mathrm{T}} + \tfrac{1}{2}\Delta\boldsymbol{S}^{\mathrm{T}}\cdot\Delta\boldsymbol{S}^{\mathrm{T}} \geq 0. \tag{35}$$

There are several ways to implement (31) and (32) in a computer program, such as the Radial Return algorithm [17], the Secant Stiffness algorithm [14,16], the Tangent Stiffness algorithm [7], the Tangent Predictor–Radial Return algorithm [6,15], and the Modified Tangent Predictor–Radial Return algorithm [3], which satisfies automatically the yield condition at the end of an increment for elastic–perfectly plastic materials and for linear hardening materials. In principle, the different methods approximate the plastic flow direction described by (25), and thus the plastic flow factor (31) and the incremental constitutive law (32), differently. Specifically, the deviatoric stress tensor $\boldsymbol{S}$ is replaced, for the Radial Return method by $\boldsymbol{S}^{\mathrm{T}}$, for the Secant Stiffness method by $\boldsymbol{S}^{\mathrm{M}}$, and for the Tangent Stiffness method and the Tangent Predictor–Radial Return method by $\boldsymbol{S}^{\mathrm{C}}$. The latter two methods are thus seen to be based directly on the tangent stiffness at the contact stress state.

From (31) it can be seen that negative plastic flow occurs whenever the inner product $2\mu \boldsymbol{S} \cdot \Delta \boldsymbol{e}$ becomes negative. Note that

$$2\mu \boldsymbol{S} \cdot \Delta \boldsymbol{e} = \boldsymbol{S} \cdot \Delta \boldsymbol{S}^{\mathrm{T}}, \tag{36}$$

which, when combined with (35), can be used to achieve the following inequalities:

$$\boldsymbol{S}^{\mathrm{T}} \cdot \Delta \boldsymbol{S}^{\mathrm{T}} \geq 0, \tag{37}$$

$$\boldsymbol{S}^{\mathrm{M}} \cdot \Delta \boldsymbol{S}^{\mathrm{T}} \geq 0. \tag{38}$$

Hence, it can be concluded that negative plastic flow will never appear with the use of the Radial Return and the Secant Stiffness methods, even in cases where it happens to the methods based on the tangent stiffness. And in such cases,

$$\boldsymbol{S}^{\mathrm{C}} \cdot \Delta \boldsymbol{S}^{\mathrm{T}} < 0, \tag{39}$$

which means that the end point of the vector $\Delta \boldsymbol{S}^{\mathrm{T}}$, based at the stress point $\boldsymbol{S}^{\mathrm{C}}$, is below the tangent plane to the yield surface at that stress point in the stress space. As explained earlier, since the yield surface is convex, such situations can arise only when the initial stress state is previously yielded and is taken to be $\boldsymbol{S}^{\mathrm{C}}$. On the other hand, when (39) is indeed encountered, it can be seen from Fig. 1(c) that elastic unloading actually occurs before the stress state is again plastically loaded. Hence, the secant stiffness method and the radial return method, although they do not produce negative plastic flow, mistakenly treat the elastic unloading as part of the plastic flow, which will also result in unpredictable errors. This observation holds for all those stress integration algorithms which do not produce negative plastic flow, whether explicit, semi-implicit or fully implicit, so long as the loading case indicated in Fig. 1(c) is not detected and modified.

Cause of negative plastic flow

Thus far we have seen that negative plastic flow occurs when numerically the inner product of the yield surface gradient tensor and the trial stress increment tensor at the reference stress point is negative, and this can happen only when the initial stress tensor is previously yielded and is taken to be the reference stress state, as shown in Fig. 1(c). It is observed that such a situation actually involves elastic unloading during the strain increment, and that this elastic unloading is mistakenly treated as plastic flow by those stress integration algorithms which do not produce negative plastic flow, as is demonstrated for von Mises materials obeying the associated flow rule.

But why will such occasions arise? The answer to this question is that flawed solution schemes for the determination of the transition parameter R are used. For example, when the initial stress state is previously yielded, it is widely adopted to set $R = 0$ in a solution procedure [13,2], which unfortunately includes the situations with negative plastic flow. Another commonly employed solution procedure for R is to use linear interpolation with iterative corrections [11,20]. This method, however, still gives zero values for R in the above situations.

Hence, it is seen that the phenomenon of negative plastic flow arises, not because of the discretization processes involved in the finite element computation, but because of the improperly implemented solution procedure for the determination of the transition parameter R. In other words, if the loading case shown in Fig. 1(c) can be detected and changed to the loading case shown in Fig. 1(a), negative plastic flow will never occur. Therefore, if the solution procedures for R can be properly modified such that the contact stress other than the initial one can be obtained in cases where elastic unloading appears, then the errors, whether they are due to the incorrect treatment of elastic unloading as plastic flow or due to the occurrence of negative plastic flow, can all be eliminated.

It is also worth while to point out that, when negative plastic flow happens, mere subincrementation of the strain increment will not help. This is because negative plastic flow will always occur, at least for the first subincrement, no matter how small it is. Moreover, even though plastic flow may become positive again in the subsequent subincrements due to the change of the stress state and hence of the direction of the yield surface gradient, numerical errors will still arise simply because the plastic flow being calculated may in fact be an elastic unloading. However, if the subincrements are small enough such that the first subincrement is inside the yield surface, and if a recheck of the yield condition is placed for each subincrement, the errors mentioned above can be avoided.

Nonetheless, to modify the solution procedure, the following measures are proposed to add to the existing ones. First, when the initial stress state is previously yielded and when the yield function is found to be positive when referred to the trial stress state $\boldsymbol{\sigma}^{\mathrm{T}}$, the inner product between $\boldsymbol{a}$ and $\Delta\boldsymbol{\sigma}^{\mathrm{T}}$ is checked. If the product is non-negative, R is set to zero. Otherwise, methods such as the one proposed by Marques [9] are used to determine R, which should be non-zero. On the other hand, when the initial stress state is not yielded, R should be calculated according to methods such as the method of linear interpolation with correction [11,20]. Note that for simple yield surfaces, such as for von Mises materials, R can be determined analytically.

Elasto-plastic computation for cornered yield surfaces

A point on a yield surface denoted by (1) is called a singular point or a corner point, if the gradient $\boldsymbol{a} \equiv \partial f/\partial\boldsymbol{\sigma}$ is not uniquely defined, such a yield surface will be called a cornered yield surface. Since a cornered yield surface is composed of several piece-wise smooth yield surfaces, stress computations at points away from the corner points can be treated in the same way as for smooth yield surfaces.

Elasto-plastic stress calculation at a singular yield point is a more complicated matter than at a regular point. This subject has been studied by Zienkiewicz, Valliappan, and King [20], by Nayak and Zienkiewicz [11], by Goncalves F° and Owen [4], and by Marques [9], for materials with cornered yield surfaces obeying the associated flow rule. Yet the issue of negative flow has not been addressed.

Now consider a singular point in the neighborhood of which the yield surface is the joint of a number of surfaces S_r, such that on the interior of each of these surfaces $\partial F/\partial\boldsymbol{\sigma} = \partial f/\partial\boldsymbol{\sigma}$ is uniquely defined. The limit of $\partial f/\partial\boldsymbol{\sigma}$ as the singular point is approached along any line on the surface S_r exists and will be writen as $\boldsymbol{a}_r \equiv \partial f_r/\partial\boldsymbol{\sigma}$. Following Koiter [5], the associated flow rule at this point can be expressed as

$$\mathrm{d}\boldsymbol{\epsilon}^{\mathrm{p}} = \sum_{i=1}^{n} \mathrm{d}\lambda_i \boldsymbol{a}_i, \tag{40}$$

where the sum is over those S_r's which are active during the increment, such that

$$F_i = 0, \tag{41}$$

$$\mathrm{d}F_i = \boldsymbol{a}_i \cdot \mathrm{d}\boldsymbol{\sigma} - A\, \mathrm{d}\lambda_i = 0, \tag{42}$$

for $i = 1, 2, \ldots, n$. The elastic response is then given by

$$\mathrm{d}\boldsymbol{\sigma} = \boldsymbol{C}\left(\mathrm{d}\boldsymbol{\epsilon} - \sum_{i=1}^{n} \mathrm{d}\lambda_i \boldsymbol{a}_i\right), \tag{43}$$

which will be used to calculate the stress increments once the plastic flow factors $d\lambda_i$ $(i = 1, 2, \ldots, n)$ are determined from procedures below. Substitution of (43) into (42) will yield the following set of linear algebraic equations for $d\lambda_i$ $(i = 1, 2, \ldots, n)$:

$$[B]\{d\lambda\} = \{db\}. \tag{44}$$

The vector $\{d\lambda\}$ is composed of the $d\lambda_i$'s and the matrix $[B]$ and the vector $\{db\}$ are respectively defined by

$$B_{ij} = (\boldsymbol{a}_i, \boldsymbol{a}_j) + A\delta_{ij}, \tag{45}$$

$$db_i = (\boldsymbol{a}_i, d\boldsymbol{\epsilon}), \tag{46}$$

where $i, j = 1, 2, \ldots, n$, and $(\cdot,\cdot)$ is an inner product with respect to the elasticity tensor $\boldsymbol{C}$, defined for any two symmetric second-order tensors (or two-tensors as they are usually called) $\boldsymbol{u}$ and $\boldsymbol{v}$ as

$$(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{C}\boldsymbol{v}. \tag{47}$$

It can be shown that the space of symmetric two-tensors with $(\cdot,\cdot)$ so defined is an inner product space. Furthermore, it can generally be shown that the matrix $[B]$, expressed in terms of an inner product and a constant A as in (45), is positive definite if $A > 0$. When $A = 0$, $[B]$ is positive or semi-positive definite depending on the relative magnitudes of n and the dimension, say m, of the subspace spanned by $\boldsymbol{a}_i (i = 1, 2, \ldots, n)$. Specifically, if $m = n$, then $[B]$ is positive definite. Otherwise, it is semi-positive definite.

For finite element computations, we shall regard those increments appearing above as finite increments. In order to ensure non-negative plastic flow, we propose the following screening test for the activeness of a particular yield surface S_r at a singular yield point. First, the inner product $(\boldsymbol{a}_r, d\boldsymbol{\epsilon})$ is checked for its sign. If it is negative, the corresponding yield surface will be considered as inactive.

The second test will be described first for hardening materials. What we do is for all the yield surfaces that have passed the first test, we solve (44) for $d\lambda_i (i = 1, 2, \ldots, n)$. Then we exclude those yield surfaces with negative $d\lambda$ values, and the previous step is repeated until all $d\lambda$ values are positive. Note that for hardening materials, the material parameter A is positive. Hence the matrix $[B]$ is positive definite as well as non-singular. Thus equation (44) always has a unique solution. Moreover, due to the positive definiteness of the matrix $[B]$, it can be proved that the second screening process can be continued to obtain at least one non-negative $d\lambda$ value.

The numerical process for non-hardening materials (i.e. for $A = 0$) depends on several factors. As we know, the dimension of the symmetric two-tensor space is six, and so is the dimension of the space where the yield surfaces are defined. When more than six yield surfaces are active at a singular point, the matrix $[B]$ will be semi-positive definite. Yet for yield functions expressible in terms of the three principal stresses, or equivalently the three stress invariants, the dimension of the space for the yield surfaces becomes three. Likewise, when more than three yield surfaces are active, $[B]$ becomes semi-positive definite. In such cases, $[B]$ is singular, and equation (44) has no unique solution, which means non-uniqueness of the Koiter's flow rule representation. Accordingly, the plastic strain increment has to be approximated arbitrarily, for example, by taking the average or the sum of those for each active yield surface, as if they were smooth. The physical meaning of this is not clear, however.

However, when the matrix $[B]$ is indeed positive definite for a non-hardening solid, the procedures for hardening solids can be applied. This is the case when the number of active yield surfaces is smaller than or equal to three for materials whose yield surfaces are expressible in terms of the principal stresses, or otherwise when it is smaller than or equal to six. This is because in such cases the gradient tensors to the active yield surfaces at the singular point are linearly independent, and hence the dimension of the subspace spanned by those gradient tensors equals the number of active yield surfaces, due to the convexity of the yield surfaces.

Summary

The phenomenon of negative plastic flow and its prevention in elasto-plastic finite element computations are investigated in detail, which can be summarized as follows:

(1) With the existing stress calculation procedures, negative plastic flow can occur when plastic stress integration is based on the tangent stiffness algorithm.

(2) When numerical errors due to the occurrence of negative plastic flow arise with the use of tangent stiffness based stress integration algorithms, numerical errors due to incorrectly treating elastic unloading as plastic flow will also occur to algorithms with positive plastic flow.

(3) Negative plastic flow occurs not because of spatial or temporal discretization errors, but because of flawed solution procedures for the determination of the transition parameter R, when the initial stress state is previously yielded.

(4) Negative plastic flow can be avoided with the proposed modifications to the existing solution procedures for R.

(5) A general and numerically efficient approach for stress calculation at singular yield points is proposed, which is consistent with the Koiter's flow rule. It is shown that when the number of active yield surfaces at a corner point is larger than the dimension of the definition-space of the yield surfaces, the representation of the Koiter's flow rule is not unique.

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